

THE HIGHLY CONNECTED EVEN-CYCLE AND EVEN-CUT MATROIDS

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ABSTRACT. The classes of even-cycle matroids, even-cycle matroids with a blocking pair, and even-cut matroids each have hundreds of excluded minors. We show that the number of excluded minors for these classes can be drastically reduced if we consider in each class only the highly connected matroids of sufficient size.

1. INTRODUCTION

An *even-cycle matroid* is a binary matroid that can be represented by a matrix with a row whose removal results in a matrix representing a graphic matroid. The complete list of excluded minors for the class of even-cycle matroids is currently unknown. Irene Pivotto and Gordon Royle (personal communication) have found nearly 400 different excluded minors. We will show that if a binary matroid M is sufficiently highly connected and has sufficient size, then to ensure that M is an even-cycle matroid, it suffices to exclude three minors: the matroid $PG(3, 2) \setminus e$ obtained by deleting one element from $PG(3, 2)$, the complete cographic matroid $M^*(K_6)$, and L_{11} , which we will define in Section 3. Let $EX(\mathcal{F})$ denote the class of matroids with no minor isomorphic to a member of \mathcal{F} . We will prove the following theorem in Section 3.

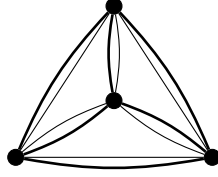
Theorem 1.1. *There exist $k, l \in \mathbb{Z}_+$ such that a vertically k -connected binary matroid with at least l elements is in $EX(PG(3, 2) \setminus e, M^*(K_6), L_{11})$ if and only if it is an even-cycle matroid.*

To state our second result, we need some more terminology. An even-cycle matroid is a binary matroid of the form $M = M\binom{w}{D}$, where $D \in GF(2)^{V \times E}$ is the vertex-edge incidence matrix of a graph $G = (V, E)$ and $w \in GF(2)^E$ is the characteristic vector of a set $W \subseteq E$.

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FIGURE 1. Even-cycle representaion of $PG(3, 2) \setminus L$

The pair (G, W) is an *even-cycle representation* of M . The edges in W are called *odd* edges, and the other edges are *even* edges. *Resigning* at a vertex u of G occurs when all the edges incident with u are changed from even to odd and vice-versa. It is easy to see that this corresponds to adding the row of the matrix corresponding to u to the characteristic vector of W . Therefore, resigning at a vertex does not change an even-cycle matroid. A pair of vertices u, v of G is a *blocking pair* if (G, W) can be resigned so that every odd edge is incident with u or v .

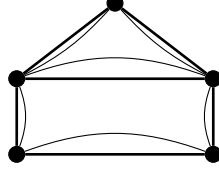
We also prove an analogue of Theorem 1.1 for the class of even-cycle matroids with a blocking pair. Let $PG(3, 2) \setminus L$ be the unique matroid obtained from $PG(3, 2)$ by deleting the three points of a line. One can check that $PG(3, 2) \setminus L$ is the even-cycle matroid represented by the graph in Figure 1, with odd edges printed in bold. We will prove the following theorem in Section 3.

Theorem 1.2. *There exist $k, l \in \mathbb{Z}_+$ such that a vertically k -connected binary matroid with at least l elements is in $EX(PG(3, 2) \setminus L, M^*(K_6))$ if and only if it is an even-cycle matroid with a blocking pair.*

An *even-cut matroid* is a matroid M that can be represented by a binary matrix with a row whose removal results in a matrix representing a cographic matroid. The situation with even-cut matroids is similar to that of even-cycle matroids: the excluded minors are currently unknown. However, we will show that if a binary matroid M is sufficiently highly-connected and has sufficient size, then to ensure that M is an even-cut matroid, it suffices to exclude two minors – the complete graphic matroid $M(K_6)$, and H_{12}^* , where H_{12} is the even-cycle matroid represented by the graph in Figure 2. Again, odd edges are printed in bold. We will prove the following theorem in Section 4.

Theorem 1.3. *There exist $k, l \in \mathbb{Z}_+$ such that a cyclically k -connected binary matroid with at least l elements is in $EX(M(K_6), H_{12}^*)$ if and only if it is an even-cut matroid.*

The work in this paper builds on our previous work in [3], which in turn builds on the work of Geelen, Gerards, and Whittle in [1]. The

FIGURE 2. Even-cycle representaion of H_{12}

results in [1] rely on the Matroid Structure Theorem by these same authors [2]. The proofs of these results are forthcoming. In Section 2, we will review the necessary definitions and results found in [1] and [3]. Section 3 gives the proofs of Theorems 1.1 and 1.2. Section 4 gives the proof of 1.3. The proofs in this paper involve some extensive computations, for which we used the SageMath software system [7].

2. PRELIMINARIES

The following notation will be used throughout this paper. We denote an empty matrix by $[\emptyset]$. We denote a group of one element by $\{0\}$ or $\{1\}$, if it is an additive or multiplicative group, respectively. We identify the additive group of 1×1 matrices over a field \mathbb{F} with \mathbb{F} itself. If S' is a subset of a set S and G is a subgroup of the additive group \mathbb{F}^S , we denote by $G|S'$ the projection of G into $\mathbb{F}^{S'}$. Similarly, if $\bar{x} \in G$, we denote the projection of \bar{x} into $\mathbb{F}^{S'}$ by $\bar{x}|S'$. Any unexplained notation and terminology will follow Oxley [4].

In this section, we recall the necessary definitions and results from [1] and [3]. Let A be a matrix over a field \mathbb{F} . Then A is a *frame matrix* if each column of A has at most two nonzero entries. We let \mathbb{F}^\times denote the multiplicative group of \mathbb{F} . Let Γ be a subgroup of \mathbb{F}^\times . A Γ -frame matrix is a frame matrix A such that:

- Each column of A with a nonzero entry contains a 1.
- If a column of A has a second nonzero entry, then that entry is $-\gamma$ where $\gamma \in \Gamma$.

In the case where Γ is the multiplicative group of one element, a matrix is a Γ -frame matrix if and only if it is the incidence matrix of a graph, with possibly a row removed. In particular, a binary matroid is graphic if and only if it can be represented by a matrix over $GF(2)$ in which no column has more than two nonzero entries.

The structure theorem given by Geelen, Gerards, and Whittle [1] is somewhat technical. They introduced the concept of a template in order to facilitate the description of the theorem. Let \mathbb{F} be a finite field.

		Z	Y_0	Y_1	C
X	0	0	A_1		
D	columns from Λ				
	Γ -frame matrix	unit columns	rows from Δ		

FIGURE 3.

A *frame template* over \mathbb{F} is a tuple $\Phi = (\Gamma, C, D, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ such that the following hold:

- (i) Γ is a subgroup of \mathbb{F}^\times .
- (ii) C, D, X, Y_0 and Y_1 are disjoint finite sets.
- (iii) $A_1 \in \mathbb{F}^{(X \cup D) \times (Y_0 \cup Y_1 \cup C)}$.
- (iv) Λ is a subgroup of the additive group of \mathbb{F}^D and is closed under scaling by elements of Γ .
- (v) Δ is a subgroup of the additive group of $\mathbb{F}^{Y_0 \cup Y_1 \cup C}$ and is closed under scaling by elements of Γ .

Let $\Phi = (\Gamma, C, D, X, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a frame template. Let E be a finite set, let $B \subseteq E$, and let $A' \in \mathbb{F}^{B \times (E-B)}$. We say that A' *respects* Φ if the following hold:

- (i) $X, D \subseteq B$ and $Y_0, Y_1, C \subseteq E - B$.
- (ii) $A'[X \cup D, Y_0 \cup Y_1 \cup C] = A_1$ and $A'[X, E - (B \cup Y_0 \cup Y_1 \cup C)] = 0$.
- (iii) There exists a set $Z \subseteq E - (B \cup Y_0 \cup Y_1 \cup C)$ such that $A'[D, Z] = 0$, each column of $A'[B - (X \cup D), Z]$ is a unit vector, and $A'[B - (X \cup D), E - (B \cup Z \cup Y_0 \cup Y_1 \cup C)]$ is a Γ -frame matrix.
- (iv) Each column of $A'[D, E - (B \cup Z \cup Y_0 \cup Y_1 \cup C)]$ is contained in Λ .
- (v) Each row of $A'[B - (X \cup D), Y_0 \cup Y_1 \cup C]$ is contained in Δ .

Figure 3 shows the structure of A' .

Suppose that A' respects Φ and that Z satisfies (iii) above. Now suppose that $A \in \mathbb{F}^{B \times (E-B)}$ satisfies the following conditions:

- (i) $A[B, E - (B \cup Z)] = A'[B, E - (B \cup Z)]$
- (ii) For each $i \in Z$ there exists $j \in Y_1$ such that the i -th column of A is the sum of the i -th and the j -th columns of A' .

We say that any such matrix *conforms* to Φ .

Let M be a matroid representable over \mathbb{F} . We say that M *conforms* to Φ if there is a matrix A that conforms to Φ such that M is isomorphic

to the vector matroid of $M([I, A])/C \setminus ((B - X) \cup Y_1)$, where I is the identity matrix with rows and columns indexed by B .

Let $\mathcal{M}(\Phi)$ denote the set of matroids representable over \mathbb{F} that conform to Φ . Recall that a matroid M is *vertically k -connected* if, for each partition (X, Y) of the ground set of M with $r(X) + r(Y) - r(M) < k - 1$, either X or Y is spanning. Also, recall that every finite field has a unique subfield of prime order; we denote that subfield by $\mathbb{F}_{\text{prime}}$. The following theorem is a slight modification of the structure theorem found in [1]. In that paper, there is no mention of the requirement that a matroid have size at least l . However, Geelen (personal communication) has stated that this is necessary to ensure that adding a finite number of matroids to the class \mathcal{M} does not add any templates to the list $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$. The proof of this theorem will be written in a future paper by Geelen, Gerards, and Whittle.

Theorem 2.1. *Let \mathbb{F} be a finite field, let m be a positive integer, and let \mathcal{M} be a minor-closed class of matroids representable over \mathbb{F} . Then there exist $k, l \in \mathbb{Z}_+$ and frame templates $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ such that*

- \mathcal{M} contains each of the classes $\mathcal{M}(\Phi_1), \dots, \mathcal{M}(\Phi_s)$,
- \mathcal{M} contains the duals of the matroids in each of the classes $\mathcal{M}(\Psi_1), \dots, \mathcal{M}(\Psi_t)$, and
- if M is a simple vertically k -connected member of \mathcal{M} with at least l elements and with no $PG(m - 1, \mathbb{F}_{\text{prime}})$ minor, then either M is a member of at least one of the classes $\mathcal{M}(\Phi_1), \dots, \mathcal{M}(\Phi_s)$, or M^* is a member of at least one of the classes $\mathcal{M}(\Psi_1), \dots, \mathcal{M}(\Psi_t)$.

The definitions and results in the remainder of this section are from [3]. There, proofs of the results can be found.

Definition 2.2. A matroid M *coconforms* to a template Φ if its dual M^* conforms to Φ .

We will use $\mathcal{M}^*(\Phi)$ to denote the set of matroids representable over \mathbb{F} that coconform to Φ . To simplify the proofs in [3], it was helpful to expand the concept of conforming slightly.

Definition 2.3. Let A' be a matrix that respects Φ , as defined above, except that we allow columns of $A'[B - (D \cup X), Z]$ to be either unit columns or zero columns. Let A be a matrix that is constructed from A' as described above. Thus, $A[B, E - (B \cup Z)] = A'[B, E - (B \cup Z)]$, and for each $i \in Z$ there exists $j \in Y_1$ such that the i -th column of A is the sum of the i -th and the j -th columns of A' . Let M be isomorphic to the vector matroid of $M([I, A])/C \setminus ((B - X) \cup Y_1)$, where I is the

identity matrix with rows and columns indexed by B . We say that A and M *virtually conform* to Φ and that A' *virtually respects* Φ . If M^* virtually conforms to Φ , we say that M *virtually coconforms* to Φ .

We will denote the set of matroids representable over \mathbb{F} that virtually conform to Φ by $\mathcal{M}_v(\Phi)$ and the set of matroids representable over \mathbb{F} that virtually coconform to Φ by $\mathcal{M}_v^*(\Phi)$.

Definition 2.4. A *reduction* is an operation on a frame template Φ that produces a frame template Φ' such that $\mathcal{M}(\Phi') \subseteq \mathcal{M}(\Phi)$.

Proposition 2.5. *The following operations are reductions on a frame template Φ :*

- (1) *Replace Γ with a proper subgroup.*
- (2) *Replace Λ with a proper subgroup closed under multiplication by elements of Γ .*
- (3) *Replace Δ with a proper subgroup closed under multiplication by elements of Γ .*
- (4) *Remove an element y from Y_1 . (More precisely, replace A_1 with $A_1[D \cup X, Y_0 \cup (Y_1 - y) \cup C]$ and replace Δ with $\Delta|(Y_0 \cup (Y_1 - y) \cup C)$).*
- (5) *For all matrices A' respecting Φ , perform an elementary row operation on $A'[D, E - B]$. (Note that this alters the matrix A_1 and performs a change of basis on Λ .)*
- (6) *If there is some element $d \in D$ such that, for every matrix A' respecting Φ , we have that $A'[\{d\}, E - B]$ is a zero row vector, remove d from D . (This simply has the effect of removing a zero row from every matrix conforming to Φ .)*
- (7) *Let $c \in C$ be such that $A_1[D \cup X, \{c\}]$ is a unit column whose nonzero entry is in the row indexed by $d \in D$, and let either $\lambda_d = 0$ for each $\lambda \in \Lambda$ or $\delta_c = 0$ for each $\delta \in \Delta$. Let Δ' be the result of adding $-\delta_c A_1[\{d\}, Y_0 \cup Y_1 \cup C]$ to each element $\delta \in \Delta$. Replace Δ with Δ' , and then remove c from C and d from D . (More precisely, replace A_1 with $A_1[(D - d) \cup X, Y_0 \cup Y_1 \cup (C - c)]$, replace Λ with $\Lambda|(D - d)$, and replace Δ with $\Delta'|(Y_0 \cup Y_1 \cup (C - c))$.)*
- (8) *Let $c \in C$ be such that $A_1[D \cup X, \{c\}]$ is a zero column and $\delta_c = 0$ for all $\delta \in \Delta$. Then remove c from C . (More precisely, replace A_1 with $A_1[D \cup X, Y_0 \cup Y_1 \cup (C - c)]$, and replace Δ with $\Delta|(Y_0 \cup Y_1 \cup (C - c))$.)*

The operations listed in the definition below are not reductions as defined above, but we continue the numbering from Proposition 2.5 for ease of reference.

Definition 2.6. A template Φ' is a *template minor* of Φ if Φ' is obtained from Φ by repeatedly performing the following operations:

- (9) Performing a reduction of type 1-8 on Φ .
- (10) Removing an element x from X and replacing A_1 with $A_1[(X - x) \cup D, Y_0 \cup Y_1 \cup C]$. (This has the effect of contracting x from every matroid conforming to Φ .)
- (11) Removing an element y from Y_0 , replacing A_1 with $A_1[X \cup D, (Y_0 - y) \cup Y_1 \cup C]$, and replacing Δ with $\Delta|((Y_0 - y) \cup Y_1 \cup C)$. (This has the effect of deleting y from every matroid conforming to Φ .)
- (12) Let $d \in D$ with $\lambda_d = 0$ for every $\lambda \in \Lambda$, and let $y \in Y_0$ be such that $(A_1)_{d,y} \neq 0$. Then contract y from every matroid conforming to Φ . (More precisely, perform row operations on A_1 so that $A_1[D \cup X, \{y\}]$ is a unit column with $(A_1)_{d,y} = 1$. Then replace every element $\delta \in \Delta$ with the row vector $-\delta_y A_1[\{d\}, Y_0 \cup Y_1 \cup C] + \delta$. This induces a group homomorphism $\Delta \rightarrow \Delta'$, where Δ' is also a subgroup of the additive group of $\mathbb{F}^{C \cup Y_0 \cup Y_1}$ and is closed under scaling by elements of Γ . Finally, replace A_1 with $A_1[D - d, (Y_0 - y) \cup Y_1 \cup C]$, project Λ into \mathbb{F}^{D-d} , and project Δ' into $\mathbb{F}^{(Y_0 - y) \cup Y_1 \cup C}$. The resulting groups play the roles of Λ and Δ , respectively in Φ' .)
- (13) Let $y \in Y_0$ with $\delta_y = 0$ for every $\delta \in \Delta$, and let either $X = \emptyset$ or $A_1[X, \{y\}]$ be a zero vector. Then contract y from every matroid conforming to Φ . (More precisely, if $A_1[D \cup X, \{y\}]$ is a zero vector, this is the same as simply removing y from Y_0 . Otherwise, choose some $d \in D$ such that $(A_1)_{d,y} \neq 0$. Then for every matrix A' that respects Φ , perform row operations so that $A_1[D \cup X, \{y\}]$ is a unit column with $(A_1)_{d,y} = 1$. This induces a group isomorphism $\Lambda \rightarrow \Lambda'$ where Λ' is also a subgroup of the additive group of \mathbb{F}^D and is closed under scaling by elements of Γ . Finally, replace A_1 with $A_1[D - d, (Y_0 - y) \cup Y_1 \cup C]$, project Λ' into \mathbb{F}^{D-d} , and project Δ into $\mathbb{F}^{(Y_0 - y) \cup Y_1 \cup C}$. The resulting groups play the roles of Λ and Δ , respectively in Φ' .)

Let Φ' be a template minor of Φ , and let A' be a matrix that virtually respects Φ' . Let A be a matrix that virtually conforms to Φ' , and let M be a matroid that virtually conforms to Φ' . We say that A' *weakly respects* Φ and that A and M *weakly conform* to Φ . Let $\mathcal{M}_w(\Phi)$ denote the set of matroids representable over \mathbb{F} that weakly conform to Φ , and let $\mathcal{M}_w^*(\Phi)$ denote the set of matroids representable over \mathbb{F} whose duals weakly conform to Φ . If $M \in \mathcal{M}_w^*(\Phi)$, we say that M *weakly coconforms* to Φ .

Lemma 2.7. *If a matroid M weakly conforms to a template Φ , then M is a minor of a matroid that conforms to Φ .*

Since Theorem 2.1 deals with minor-closed classes, it can be generalized using the preceding lemma.

Corollary 2.8. *Let \mathbb{F} be a finite field, let m be a positive integer, and let \mathcal{M} be a minor-closed class of matroids representable over \mathbb{F} . Then there exist $k, l \in \mathbb{Z}_+$ and frame templates $\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t$ such that*

- \mathcal{M} contains each of the classes $\mathcal{M}_w(\Phi_1), \dots, \mathcal{M}_w(\Phi_s)$,
- \mathcal{M} contains the duals of the matroids in each of the classes $\mathcal{M}_w(\Psi_1), \dots, \mathcal{M}_w(\Psi_t)$, and
- if M is a simple vertically k -connected member of \mathcal{M} with at least l elements and with no $PG(m-1, \mathbb{F}_{\text{prime}})$ minor, then either M is a member of at least one of the classes $\mathcal{M}_v(\Phi_1), \dots, \mathcal{M}_v(\Phi_s)$ or M^* is a member of at least one of the classes $\mathcal{M}_v(\Psi_1), \dots, \mathcal{M}_v(\Psi_t)$.

Suppose that a matroid M weakly conforms to a frame template Φ if and only if it weakly conforms to a frame template Φ' . Then we say that Φ is *equivalent* to Φ' and write $\Phi \sim \Phi'$. It is clear that \sim is indeed an equivalence relation.

Definition 2.9. Let $T_{\mathbb{F}}$ be the set of all frame templates over \mathbb{F} . We define a preorder \preceq on $T_{\mathbb{F}}$ as follows. We say $\Phi \preceq \Phi'$ if $\mathcal{M}_w(\Phi) \subseteq \mathcal{M}_w(\Phi')$. This is indeed a preorder since reflexivity and transitivity follow from the subset relation.

Let Φ_0 be the frame template with all groups trivial and all sets empty. We call this template the *trivial template*. In general, we say that a template Φ is *trivial* if $\Phi \preceq \Phi_0$. It is easy to see that for any template Φ , we have $\Phi_0 \preceq \Phi$. Therefore, if $\Phi \preceq \Phi_0$, then actually $\Phi \sim \Phi_0$.

Lemma 2.10. *Every frame template Φ is equivalent to a frame template Φ' with $X = \emptyset$.*

We proceed to describe a collection of minimal nontrivial templates.

Definition 2.11. Let Φ_C be the template with all groups trivial and all sets empty except that $|C| = 1$ and $\Delta \cong GF(2)$. Let Φ_D be the template with all groups trivial and all sets empty except that $|D| = 1$ and $\Lambda \cong GF(2)$. Let Φ_{Y_0} be the template with all groups trivial and all sets empty except that $|Y_0| = 1$ and $\Delta \cong GF(2)$. Let Φ_{CD} be the template with $X = Y_0 = Y_1 = \emptyset$, with $|C| = |D| = 1$, with

		Z	Y_0	Y_1	C_0	C_1
D_0	columns from ΛD_0	0	*		I	*
D_1	columns from ΛD_1	0			0	
Γ -frame matrix		unit or zero columns	rows from Δ			

FIGURE 4. Φ -standard form

$\Delta \cong \Lambda \cong GF(2)$, with Γ trivial, and with $A_1 = [1]$. Let Φ_{Y_1} be the template with all groups trivial, with $X = C = Y_0 = \emptyset$, with $|Y_1| = 3$ and $|D| = 2$, and with $A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Because the proof of following lemma makes use of operations other than those we defined in Proposition 2.5 and Definition 2.6, we choose to make only occasional use of it.

Lemma 2.12. *The following relations hold:*

- (1) $\Phi_{Y_1} \preceq \Phi_D$
- (2) $\Phi_{Y_1} \preceq \Phi_C$
- (3) $\Phi_{Y_0} \preceq \Phi_C$
- (4) $\Phi_C \preceq \Phi_{CD}$
- (5) $\Phi_D \preceq \Phi_{CD}$

Lemma 2.13. *Let Φ be a template with $y \in Y_1$. Let Φ' be the template obtained from Φ by removing y from Y_1 and placing it in Y_0 . Then $\Phi' \preceq \Phi$.*

We call the operation described in Lemma 2.13 a *y-shift*.

Let $\Phi = (\{1\}, C, D, \emptyset, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a frame template, where we have assumed that $X = \emptyset$ by Lemma 2.10. Choose a basis C_0 for $M[A_1[D, C]]$. Let A' be a matrix that virtually respects Φ and is in the form shown in Figure 4, with the stars representing arbitrary binary matrices, with $D = D_0 \cup D_1$, and with $C = C_0 \cup C_1$. We will say that A' is in a Φ -standard form.

Note that any matrix virtually respecting Φ is row equivalent to a matrix in a Φ -standard form; therefore, throughout the rest of this paper, we will assume that all matrices virtually respecting a frame template Φ are in a Φ -standard form. This necessarily implies that we are assuming that $X = \emptyset$. Also, the operations (1)-(13) to which we

will refer throughout the rest of this paper are the operations (1)-(8) from Proposition 2.5 and (9)-(13) from Definition 2.6.

Lemma 2.14. *If $\Phi = (\{1\}, C, D, \emptyset, Y_0, Y_1, A_1, \Delta, \Lambda)$ is a binary frame template with $\Lambda|_{D_1}$ nontrivial, then $\Phi_D \preceq \Phi$.*

Lemma 2.15. *If Φ is a frame template with $X = \emptyset$ and with Δ trivial, then Φ is equivalent to a template Φ' where $A_1[D, Y_1]$ is a matrix with every column nonzero and where no column is a copy of another. Moreover, if Φ is a binary frame template, then $M[A_1[D, Y_1]]$ is simple.*

Lemma 2.16. *Let $\mathcal{T}_{EX(PG(3,2))} = \{\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t\}$. If $\Phi \in \{\Phi_1, \dots, \Phi_s\}$, then either $\Phi = \Phi_D$ or*

$$\Phi = (\{1\}, \emptyset, D, \emptyset, Y_0, Y_1, A_1, \{0\}, \{0\}),$$

for some matrix A_1 and some sets D , Y_0 , and Y_1 .

Definition 2.17. Let X_r be the largest simple matroid of rank r that virtually conforms to Φ_{Y_1} .

Note that $X_1 = U_{1,1}$, and if $r \geq 2$, then X_r is the vector matroid of the following binary matrix, where we choose for the Γ -frame matrix the matrix representation of $M(K_{r-1})$, so that the identity matrices are $(r-2) \times (r-2)$ matrices. We will call this matrix A_r :

0	1 0 1	1...1	0...0	1...1
	0 1 1	0...0	1...1	1...1
Γ -frame matrix	0	I	I	I

Lemma 2.18. *The class $\mathcal{M}_v(\Phi_{Y_1})$ is the class of even-cycle matroids with a blocking pair. This class is minor-closed.*

3. EVEN-CYCLE MATROIDS

In this section, we prove Theorems 1.1 and 1.2. We define L_{11} to be the vector matroid of the following binary matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Any even-cycle matroid M can be represented by a binary matrix with a row whose removal results in a matrix representing a graphic matroid. Thus, there is some binary extension N of M on ground set $E(M) \cup \{e\}$ such that N/e is graphic. Thus, to check if a binary

matroid M is even-cycle, it suffices to check if N/e is graphic for any binary extension N of M . If this is not the case for any such N , then M is not even-cycle. We use the mathematics software system SageMath to assist in these computations. The SageMath worksheet where we performed these computations can be found in Appendix A. Lemmas A.1-A.22 can also be found in Appendix A. Let $PG(3, 2) \setminus e$ be the unique matroid obtained from $PG(3, 2)$ by deleting one element.

Theorem 3.1. *Each of the matroids $PG(3, 2) \setminus e$, $M^*(K_6)$, and L_{11} is an excluded minor for the class of even-cycle matroids.*

Proof. It is easily verified that the growth rate for the class of even-cycle matroids is $2\binom{r}{2} + 1 = r^2 - r + 1$. Therefore, the matroid $PG(3, 2) \setminus e$, which has rank 4 and size 14 is too large to be even-cycle. By symmetry, deletion of any element from $PG(3, 2) \setminus e$ results in the unique matroid (up to isomorphism) obtained from $PG(3, 2)$ by deleting two elements. This is exactly the largest simple even-cycle matroid of rank 4. Thus, deletion of any element from $PG(3, 2) \setminus e$ results in an even-cycle matroid. To see that contraction of any element from $PG(3, 2) \setminus e$ results in an even-cycle matroid, note that any binary matroid of rank 3 is even-cycle since removal of any row results in a matrix that obviously has at most two nonzero entries per column.

The fact that $M^*(K_6)$ and L_{11} are excluded minors for the class of even-cycle matroids was verified using SageMath in Lemmas A.1 and A.2. \square

Theorem 3.1 shows that the class of even-cycle matroids is contained in $EX(PG(3, 2) \setminus e, M^*(K_6), L_{11})$. We will prove Theorem 1.1, which shows that for sufficiently highly connected binary matroids, the reverse inclusion holds. First, we prove several lemmas. The conclusions of the lemmas in this section are stated up to reordering of the rows and columns of $A_1[D, Y_1]$.

Lemma 3.2. *Let Φ be a binary frame template of the form*

$$\Phi = (\{1\}, \emptyset, D, \emptyset, Y_0, Y_1, A_1, \{0\}, \{0\}).$$

Then at least one of the following holds:

- (1) *There exist $k, l \in \mathbb{Z}_+$ such that no simple, vertically k -connected matroid with at least l elements virtually conforms to Φ , or*
- (2) *A_1 is of the following form, with $Y_1 = Y'_1 \cup Y''_1$ and each P_i an arbitrary binary matrix:*

$$\begin{array}{|c|c|c|} \hline Y'_1 & Y''_1 & Y_0 \\ \hline I & P_1 & P_0 \\ \hline \end{array}$$

Proof. Suppose (1) does not hold. By operation (5), we may assume that A_1 is of the following form, with $Y_0 = V_0 \cup V_1$, with $Y_1 = Y'_1 \cup Y''_1$ and with each P_i an arbitrary binary matrix:

$$\begin{array}{c|c|c|c} Y'_1 & Y''_1 & V_0 & V_1 \\ \hline I & P_1 & 0 & P_0 \\ \hline 0 & 0 & I & P_2 \end{array}.$$

Let $r' = r(M[P_0])$. For any matroid M virtually conforming to Φ , let r and λ be the rank and connectivity functions of M , respectively. We have $r(Y_0) = |V_0| + r'$, and $r(E(M) - Y_0) = r(M) - |V_0|$. Thus, $\lambda(Y_0) = r'$. Let $k > r' + 1$. Then M is not vertically k -connected unless either Y_0 or $E(M) - Y_0$ is spanning. If Y_0 is spanning and M is simple, then the Γ -frame matrix must be a 0×0 matrix. This implies that $|E(M)| \leq |Y_0 \cup Y_1|$. Let $l > |Y_0 \cup Y_1|$. Then no simple, vertically k -connected matroid with at least l elements conforms to Φ unless $E(M) - Y_0$ is spanning. This implies that $V_0 = \emptyset$ and therefore, (2) holds. \square

Lemma 3.3. *Let Φ be of the form*

$$\Phi = (\{1\}, \emptyset, D, \emptyset, Y_0, Y'_1 \cup Y''_1, A_1, \{0\}, \{0\})$$

with A_1 of the form given in conclusion (2) of Lemma 3.2. Then at least one of the following holds:

(1) $\mathcal{M}_w(\Phi) \not\subseteq EX(PG(3, 2) \setminus e)$,

(2) P_1 is a restriction of a matrix of the form $\begin{bmatrix} 1 \cdots 1 \\ I \end{bmatrix}$ and P_0 is of the following form, where each column of $[H_1|H_0]$ has at most two nonzero entries:

$$\left[\begin{array}{c|c} 1 \cdots 1 & 0 \cdots 0 \\ \hline H_1 & H_0 \end{array} \right],$$

(3) P_1 is a restriction of a matrix of the form $\begin{bmatrix} 1 \cdots 1 \\ 1 \cdots 1 \\ I \end{bmatrix}$ and P_0 is of the following form, where each column of $H_{1,1}$ and $H_{0,0}$ has at most two nonzero entries and where each column of $H_{0,1}$ and $H_{1,0}$ is a unit column or a zero column:

$$\left[\begin{array}{c|c|c|c} 1 \cdots 1 & 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 \\ 1 \cdots 1 & 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \\ \hline H_{1,1} & H_{1,0} & H_{0,1} & H_{0,0} \end{array} \right],$$

(4) $P_1 = [1, 1, 0, \dots, 0]^T$ or $P_1 = [1, 1, 1, 0, \dots, 0]^T$, or

(5) $Y''_1 = \emptyset$.

Proof. Suppose (1) does not hold. By Lemma 2.15, $M[A_1[D, Y_1]]$ is simple. By Lemma A.3, $M[A_1[D, Y_1]]$ has no circuit of size at least 5; thus each column of P_1 has at most three nonzero entries. Therefore, every column of P_1 has exactly two or three nonzero entries. Let $P_1 = [P_{1,2}|P_{1,3}]$, where $P_{1,i}$ consists entirely of columns with exactly i nonzero entries. By Lemmas A.4 and A.5, along with the fact that $M[A_1[D, Y_1]]$ is simple, $P_{1,2}$ must be a restriction of a matrix the following form:

$$\begin{bmatrix} 1 \cdots 1 \\ I \end{bmatrix}.$$

By Lemma A.4, any pair of columns in $P_{1,3}$ must both have nonzero entries in the same two rows. Suppose a third column in $P_{1,3}$ does not have nonzero entries in the same two rows as the other two columns. It follows that $P_{1,3}$ must contain the following submatrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

However, this contains the submatrix forbidden by Lemma A.5.

Therefore, $P_{1,3}$ is of the following form: $\begin{bmatrix} 1 \cdots 1 \\ 1 \cdots 1 \\ I \end{bmatrix}$.

We will now show that either $P_{1,2}$ or $P_{1,3}$ is an empty matrix. Consider the matrix

$$P'_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

As observed in Lemma A.7, this is not a submatrix of P_1 because in the matrix $[I_4|P'_1]$, columns 1, 2, 4, 5, and 6 form a circuit of size 5. Thus, if v is a column of $P_{1,3}$, then any column w of $P_{1,2}$ must have its nonzero entries in two of the rows that contain the nonzero entries of v . Thus, if $P_{1,3}$ contains exactly one column, then P_1 is a restriction of

the following matrix: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, but this is the submatrix forbidden

by Lemma A.6. Therefore, we see that if $P_{1,3}$ is a nonempty matrix, then P_1 must be a restriction of a matrix of the following form:

$$\left[\begin{array}{c|ccc} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \hline 0 & & & \\ \vdots & & I & \\ 0 & & & \end{array} \right]$$

However, if P_1 is of this form and contains the column with two nonzero entries, then by adding the first row to the second and swapping the resulting unit column with the appropriate column of the original identity matrix, we obtain a matrix of the form: $\begin{bmatrix} 1 & \cdots & 1 \\ & I & \end{bmatrix}$.

Thus, P_1 is either of the form $\begin{bmatrix} 1 & \cdots & 1 \\ & I & \end{bmatrix}$ or of the form $\begin{bmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ & I & \end{bmatrix}$.

Suppose P_1 is of the form $\begin{bmatrix} 1 & \cdots & 1 \\ & I & \end{bmatrix}$. If (2) does not hold, then either (4) or (5) holds or $[P_1|P_0]$ contains one of the following submatrices, with the submatrix to the left of the vertical line contained in P_1 , with the column to the right of the vertical line contained in P_0 , and with either $x = 0$ or $x = 1$:

$$\left[\begin{array}{cc|c} 1 & 1 & x \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right], \left[\begin{array}{cc|c} 1 & 1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array} \right], \left[\begin{array}{cc|c} 1 & 1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array} \right].$$

By Lemma A.15, (1) holds.

Now, suppose P_1 is of the form $\begin{bmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ & I & \end{bmatrix}$. If (3) does not hold, then either (4) or (5) holds or $[P_1|P_0]$ contains one of the following submatrices, with the submatrix to the left of the vertical line contained in P_1 , with the column to the right of the vertical line contained in P_0 , and with either $x = 0$ or $x = 1$:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right], \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right], \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array} \right],$$

$$\left[\begin{array}{cc|c} 1 & 1 & x \\ 1 & 1 & x \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right], \left[\begin{array}{cc|c} 1 & 1 & x \\ 1 & 1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array} \right], \left[\begin{array}{cc|c} 1 & 1 & x \\ 1 & 1 & x \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array} \right].$$

If $[P_1|P_0]$ contains any of the first three of these submatrices, then Lemma A.16 implies that (1) holds. The last three contain submatrices forbidden by Lemma A.15; therefore (1) holds in that case as well. \square

Lemma 3.4. *Let Φ be of the form*

$$\Phi = (\{1\}, \emptyset, D, \emptyset, Y_0, Y'_1 \cup Y''_1, A_1, \{0\}, \{0\})$$

with A_1 of the form given in conclusion (2) of Lemma 3.2 and with $|Y''_1| = 1$. Then at least one of the following holds:

- (1) $\mathcal{M}_w(\Phi) \not\subseteq EX(PG(3, 2) \setminus e)$,
- (2) $P_1 = [1, 1, 0 \dots, 0]^T$ and P_0 is of the following form, where no column of $[H_{1,1}|H_{1,0}|H_{0,1}|H_{0,0}]$ has three or more nonzero entries and where at most one of $H_{1,1}$, $H_{1,0}$ and $H_{0,1}$ has columns with two nonzero entries:

$$\left[\begin{array}{c|c|c|c} 1 \dots 1 & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 \\ 1 \dots 1 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 \\ \hline H_{1,1} & H_{1,0} & H_{0,1} & H_{0,0} \end{array} \right], \text{ or}$$

- (3) $P_1 = [1, 1, 1, 0 \dots, 0]^T$ and P_0 is a restriction of a matrix of the following form, where each column of $H_{1,1,1}$, $H_{1,0,0}$, $H_{0,1,0}$, and $H_{0,0,1}$ is either a unit column or a zero column, and where each column of $H_{1,1,0}$ and $H_{0,0,0}$ has at most two nonzero entries:

$$\left[\begin{array}{c|c|c|c|c|c|c} 1 \dots 1 & 1 \dots 1 & 1 & 0 & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 & 0 \dots 0 \\ 1 \dots 1 & 1 \dots 1 & 0 & 1 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 \\ 1 \dots 1 & 0 \dots 0 & 1 & 1 & 0 \dots 0 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 \\ \hline H_{1,1,1} & H_{1,1,0} & 0 & 0 & H_{1,0,0} & H_{0,1,0} & H_{0,0,1} & H_{0,0,0} \end{array} \right].$$

Proof. Suppose (1) does not hold. Since $|Y''_1| = 1$, Lemma 3.3 implies that either $P_1 = [1, 1, 0 \dots, 0]^T$ or $P_1 = [1, 1, 1, 0 \dots, 0]^T$. First, consider the case where $P_1 = [1, 1, 0 \dots, 0]^T$. If P_0 contains either $[0, 0, 1, 1, 1]^T$ or $[1, 0, 1, 1, 1]^T$ as a submatrix, then by contracting that column of Y_0 , we obtain in P_1 a submatrix forbidden by Lemmas A.4 or A.7, respectively. This, with the fact that no column of P_0 can contain five nonzero entries, shows that $[P_1|P_0]$ is of the following form, where

each column of $[H_{1,1}|H_{1,0}|H_{0,1}|H_{0,0}]$ contains at most two nonzero entries:

$$\left[\begin{array}{c|c|c|c|c} 1 & 1 \dots 1 & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 \\ 1 & 1 \dots 1 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 \\ \hline 0 & H_{1,1} & H_{1,0} & H_{0,1} & H_{0,0} \end{array} \right].$$

By Lemma A.8 and Lemma A.17, at most one of $H_{1,1}$, $H_{1,0}$ and $H_{0,1}$ contains a column with two nonzero entries. Therefore, (2) holds.

Now consider the case where $P_1 = [1, 1, 1, 0 \dots, 0]^T$. If P_0 contains either $[0, 0, 0, 1, 1, 1]^T$ or $[1, 0, 0, 1, 1]^T$ as a submatrix, then by contracting that column of Y_0 , we obtain in P_1 a submatrix forbidden by Lemmas A.4 or A.7, respectively. This, with the fact that no column of P_0 can contain five nonzero entries, shows that $[P_1|P_0]$ is of the following form, where each column of $H_{1,1,1}$, $H_{1,0,0}$, $H_{0,1,0}$, and $H_{0,0,1}$ is either a unit column or a zero column, and where each column of $H_{1,1,0}$, $H_{1,0,1}$, $H_{0,1,1}$, and $H_{0,0,0}$ has at most two nonzero entries

$$\left[\begin{array}{c|c|c|c|c|c|c|c} 1 \dots 1 & 1 \dots 1 & 1 \dots 1 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 & 0 \dots 0 \\ 1 \dots 1 & 1 \dots 1 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 \\ 1 \dots 1 & 0 \dots 0 & 1 \dots 1 & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 \\ \hline H_{1,1,1} & H_{1,1,0} & H_{1,0,1} & H_{0,1,1} & H_{1,0,0} & H_{0,1,0} & H_{0,0,1} & H_{0,0,0} \end{array} \right].$$

By Lemma A.18, at most one of $H_{1,1,0}$, $H_{1,0,1}$, and $H_{0,1,1}$ contains any nonzero entries. Therefore, if any simple matroids virtually conform to Φ , then (3) holds. Otherwise, we discard Φ since the results in this paper deal with simple matroids. \square

Lemma 3.5. *Let Φ be of the form*

$$\Phi = (\{1\}, \emptyset, D, \emptyset, Y_0, Y'_1 \cup Y''_1, A_1, \{0\}, \{0\})$$

with A_1 of the form given in conclusion (2) of Lemma 3.2. Then at least one of the following holds:

- (1) $\mathcal{M}_w(\Phi) \not\subseteq EX(PG(3, 2) \setminus e, L_{11})$,
- (2) P_0 is of the following form, where each column of $[H_1|H_0]$ has at most two nonzero entries:

$$\left[\begin{array}{c|c} 1 \dots 1 & 0 \dots 0 \\ \hline H_1 & H_0 \end{array} \right], \text{ or}$$

- (3) P_0 is of the following form, where each column of $H_{1,1}$ and $H_{0,0}$ has at most two nonzero entries and where each column of $H_{0,1}$ and $H_{1,0}$ is a unit column or a zero column:

$$\left[\begin{array}{c|c|c|c} 1 \dots 1 & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 \\ 1 \dots 1 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 \\ \hline H_{1,1} & H_{1,0} & H_{0,1} & H_{0,0} \end{array} \right]$$

Proof. If P_0 satisfies the hypotheses of any one of Lemmas A.8-A.14, then the result holds. Otherwise, P_0 contains no column with five nonzero entries and also contains none of the following as a submatrix:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

From these, it follows easily that P_0 has at most one column, other than duplicate columns, with more than two nonzero entries. In this case, clearly (2) holds. If P_0 does contain duplicate columns, then we need not consider Φ because no simple matroid virtually conforms to Φ . \square

We now prove Theorem 1.1, which we restate below.

Theorem 3.6. *There exist $k, l \in \mathbb{Z}_+$ such that a vertically k -connected binary matroid with at least l elements is in $EX(PG(3, 2) \setminus e, M^*(K_6), L_{11})$ if and only if it is an even-cycle matroid.*

Proof. Let \mathcal{M} denote the class of binary matroids in $EX(PG(3, 2) \setminus e, M^*(K_6), L_{11})$ and let $\mathcal{T}_{\mathcal{M}} = \{\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t\}$. Consider any template $\Psi \in \{\Psi_1, \dots, \Psi_t\}$. Recall that every matroid coconforming to Ψ must be contained in the minor-closed class \mathcal{M} . Every cographic matroid is a minor of a matroid that coconforms to Ψ . Therefore, Ψ does not exist since \mathcal{M} does not contain $M^*(K_6)$. Thus, $t = 0$ and $\mathcal{T}_{\mathcal{M}} = \{\Phi_1, \dots, \Phi_s\}$. Because $PG(3, 2) \setminus e$, $M^*(K_6)$, and L_{11} are simple matroids, it suffices to consider the simple matroids conforming to these templates.

Any matroid containing $PG(3, 2)$ as a minor of course also contains $PG(3, 2) \setminus e$. Therefore, the proof of Lemma 2.16, found in [3], shows that for any template $\Phi \in \{\Phi_1, \dots, \Phi_s\}$, either $\Phi = \Phi_D$ or

$$\Phi = (\{1\}, \emptyset, D, \emptyset, Y_0, Y_1, A_1, \{0\}, \{0\})$$

for some matrix A_1 and some sets D , Y_0 , and Y_1 . We will show that in fact $\Phi = \Phi_D$ so that $\mathcal{T}_{\mathcal{M}} = \{\Phi_D\}$.

Suppose, for contradiction, that $\Phi = (\{1\}, \emptyset, D, \emptyset, Y_0, Y_1, A_1, \{0\}, \{0\})$. Since $\mathcal{M}_w(\Phi) \subseteq EX(PG(3, 2) \setminus e)$, conclusion (2) of Lemma 3.2 holds, and one of conclusions (2)-(5) of Lemma 3.3 holds. If conclusion (2) holds, then any matroid virtually conforming to Φ is clearly an even-cycle matroid. Similarly, if conclusion (3) holds, then by adding the first row to the second we see that any matroid virtually conforming to Φ is an even-cycle matroid. Therefore, if either (2) or (3) of Lemma 3.3

holds, then $\Phi \preceq \Phi_D$. Since we already know that $\mathcal{M}(\Phi_D) \subseteq \mathcal{M}$, we may discard Φ as a template that describes \mathcal{M} .

Now suppose conclusion (4) of Lemma 3.3 holds. Then, in Lemma 3.4, either (2) or (3) holds. By adding the first row to the second we see that any matroid virtually conforming to Φ is an even-cycle matroid. Therefore, we may again discard Φ as a template that describes \mathcal{M} .

Now suppose conclusion (5) of Lemma 3.3 holds; so $Y_1'' = \emptyset$. In Lemma 3.5, either conclusion (2) or (3) holds. By adding a row to another if necessary, we again see that $\Phi \preceq \Phi_D$. Thus, Φ may be discarded. \square

We now wish to prove Theorem 1.2. Let $PG(3, 2)_{-2}$ be the unique matroid obtained from $PG(3, 2)$ by deleting two elements, and let $PG(3, 2) \setminus L$ be the unique matroid obtained from $PG(3, 2)$ by deleting the three points of a line. One can check that $PG(3, 2) \setminus L$ is the even-cycle matroid represented by $\pm K_4$, the signed graph obtained from K_4 by replacing each edge with a pair of edges consisting of both an odd edge and an even edge. We will show that $PG(3, 2) \setminus L$ and $M^*(K_6)$ are excluded minors for the class of even-cycle matroids with a blocking pair. The following lemma will be used.

Lemma 3.7. *A matroid M is an even-cycle matroid with a blocking pair if and only if its cosimplification $co(M)$ also is.*

Proof. The class of even-cycle matroids with a blocking pair is minor-closed; therefore if M is such a matroid, then so is $co(M)$.

For the converse, let M be even-cycle with a blocking pair and consider an even-cycle representation of M with all odd edges incident with a vertex from the blocking pair. It suffices to consider coextensions N of M such that $N/e = M$ and such that either $\{e, f\}$ is a series pair of N or e is a coloop of N . First, we consider the case where $\{e, f\}$ is a series pair. If f is represented by an even edge in the even-cycle representation of M , then e and f in N are represented by edges obtained by subdividing f in M . This has no effect on the blocking pair. If f is represented by an odd edge other than a loop, we resign at a vertex in the blocking pair that is incident with f . This maintains the blocking pair, but now f is represented by an even edge as above. Now consider the case where f is represented by an odd loop. Since M is even-cycle with a blocking pair, recall from Lemma 2.18 that M is a restriction of a matroid represented by a matrix of the following form:

	f					
0	1	0	1	$1 \dots 1$	$0 \dots 0$	$1 \dots 1$
	0	1	1	$0 \dots 0$	$1 \dots 1$	$1 \dots 1$
Γ -frame matrix	0			I	I	I

Since N contains $\{e, f\}$ as a series pair, N is a restriction of a matroid N' represented by a matrix of the following form:

e		f				
0	0	1 0 1	1...1	0...0	1...1	
0		0 1 1	0...0	1...1	1...1	
0	Γ -frame matrix	0	I	I	I	
1	0.....0	1 0 0	0...0	0...0	0...0	

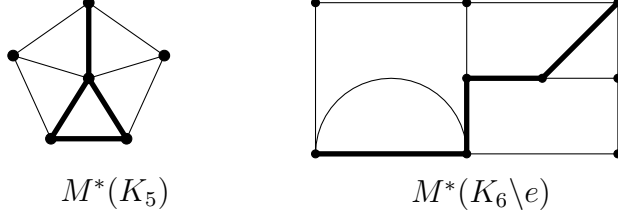
By Lemma 2.18, N' is even-cycle with a blocking pair. Therefore, so is N .

Now we consider the case where e is a coloop of N . Then N can be represented by a graph obtained from the graph representing M by adding a new vertex and joining it to any other vertex with an even edge. The blocking pair is maintained. \square

Theorem 3.8. *The matroids $PG(3, 2) \setminus L$ and $M^*(K_6)$ are excluded minors for the class of even-cycle matroids with a blocking pair.*

Proof. Recall from Definition 2.17 and Lemma 2.18 that X_r is the largest simple matroid of rank r that is even-cycle with a blocking pair. Note that X_4 is the matroid obtained from $PG(3, 2)$ by deleting an independent set of size 3. Therefore, $PG(3, 2) \setminus L$ is not a restriction of X_4 . By Lemma 2.18, $PG(3, 2) \setminus L$ is not an even-cycle matroid with a blocking pair. However, since $X_3 = PG(2, 2) = F_7$, any binary matroid of rank at most 3 is an even-cycle matroid with a blocking pair. Therefore, $PG(3, 2) \setminus L / e$ is even-cycle with a blocking pair for any element e of $PG(3, 2) \setminus L$. Moreover, by deleting any element from $PG(3, 2) \setminus L$, we obtain a restriction of X_4 . Therefore, $PG(3, 2) \setminus L$ is an excluded minor for the class of even-cycle matroids with a blocking pair.

Since $M^*(K_6)$ is not even-cycle, it remains to show that $M^*(K_6 \setminus e)$ and $M^*(K_6 / e)$ are even-cycle with a blocking pair. By Lemma 3.7, $M^*(K_6 / e)$ has an even-cycle representation with a blocking pair if and only if $M^*(K_5)$ does. One can check that $M^*(K_5)$ and $M^*(K_6 \setminus e)$ have the even-cycle representations given in Figure 5, with odd edges printed in bold. Each of these representations have blocking pairs. For the even-cycle representation of $M^*(K_5)$, note that by resigning, we may obtain the wheel \mathcal{W}_5 , with every edge odd.

FIGURE 5. Even-cycle representaions of $M^*(K_5)$ and $M^*(K_6 \setminus e)$

□

We now prove Theorem 1.2, which we restate below.

Theorem 3.9. *There exist $k, l \in \mathbb{Z}_+$ such that a vertically k -connected binary matroid with at least l elements is in $EX(PG(3, 2) \setminus L, M^*(K_6))$ if and only if it is an even-cycle matroid with a blocking pair.*

Proof. Recall that $\mathcal{M}_w(\Phi_{Y_1})$ is the class of even-cycle matroids with a blocking pair. We will show that $\{\Phi_{Y_1}\}$ is the set of templates describing $\mathcal{M}_w(\Phi) \subseteq EX(PG(3, 2) \setminus L, M^*(K_6))$.

Let Φ be a binary frame template such that $\mathcal{M}_w(\Phi) \subseteq EX(PG(3, 2) \setminus L, M^*(K_6))$. To show that $PG(3, 2) \setminus L$ is a minor of some matroid virtually conforming to a template, it suffices to show that $PG(3, 2)_{-2}$ is a minor of some matroid virtually conforming to that template. Moreover, for each computation in Appendix A where we showed that L_{11} was a minor of a matroid virtually conforming to some template, we also showed that $PG(3, 2)_{-2}$ is a minor of some matroid virtually conforming to that template. Therefore, $\mathcal{M}_w(\Phi) \subseteq EX(PG(3, 2) \setminus e, M^*(K_6), L_{11})$.

By Lemma 2.16, since $PG(3, 2) \setminus L$ conforms to Φ_D , we have that Φ is of the form

$$\Phi = (\{1\}, \emptyset, D, \emptyset, Y_0, Y_1, A_1, \{0\}, \{0\}).$$

By Lemma 3.2, we may assume that A_1 is of the following form, with $Y_1 = Y'_1 \cup Y''_1$ and each P_i an arbitrary binary matrix:

$$\begin{array}{|c|c|c|} \hline Y'_1 & Y''_1 & Y_0 \\ \hline I & P_1 & P_0 \\ \hline \end{array}$$

By Lemma 3.3 and Lemma A.19, either $P_1 = [1, 1, 0, \dots, 0]^T$, or $P_1 = [1, 1, 1, 0, \dots, 0]^T$, or $Y''_1 = \emptyset$.

Case 1: Suppose $P_1 = [1, 1, 0, \dots, 0]^T$. By Lemma 3.4, P_0 is of the following form, where no column of $[H_{1,1}|H_{1,0}|H_{0,1}|H_{0,0}]$ has three or

more nonzero entries:

$$\left[\begin{array}{c|c|c|c} 1 \cdots 1 & 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 \\ 1 \cdots 1 & 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \\ \hline H_{1,1} & H_{1,0} & H_{0,1} & H_{0,0} \end{array} \right]$$

By Lemma A.20, no column of $[H_{1,1}|H_{1,0}|H_{0,1}]$ has two or more nonzero entries. Let A be any matrix with r rows that virtually conforms to Φ . Add a row $r+1$ to the matrix, where row $r+1$ is the sum of rows $2, \dots, r$. Then one can see that $M[A]$ is an even-cycle matroid with row 1 being the sign row and with a blocking pair represented by rows 2 and $r+1$. Therefore, $\Phi \preceq \Phi_{Y_1}$, and we may discard Φ .

Case 2: Suppose $P_1 = [1, 1, 1, 0, \dots, 0]^T$. By Lemma 3.4, P_0 is of the following form, where each column of $H_{1,1,1}$, $H_{1,0,0}$, $H_{0,1,0}$, and $H_{0,0,1}$ is either a unit column or a zero column, and where each column of $H_{1,1,0}$ and $H_{0,0,0}$ has at most two nonzero entries:

$$\left[\begin{array}{c|c|c|c|c|c|c|c} 1 \cdots 1 & 1 \cdots 1 & 1 & 0 & 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 \\ 1 \cdots 1 & 1 \cdots 1 & 0 & 1 & 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 \\ 1 \cdots 1 & 0 \cdots 0 & 1 & 1 & 0 \cdots 0 & 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \\ \hline H_{1,1,1} & H_{1,1,0} & 0 & 0 & H_{1,0,0} & H_{0,1,0} & H_{0,0,1} & H_{0,0,0} \end{array} \right]$$

By Lemma A.21, $H_{1,1,0}$ must be a zero matrix. Therefore, by adding the first row to the second, we see that any matroid virtually conforming to Φ is even-cycle with a blocking pair by taking the first row to be the sign row and taking the blocking pair to be represented by the second and third rows. Therefore, $\Phi \preceq \Phi_{Y_1}$, and we may discard Φ .

Case 3: Suppose $Y_1'' = \emptyset$. Either conclusion (2) or conclusion (3) of Lemma 3.5 must hold. First, suppose (2) of 3.5 holds. Then P_0 is of the following form, where each column of $[H_1|H_0]$ has at most two nonzero entries:

$$\left[\begin{array}{c|c} 1 \cdots 1 & 0 \cdots 0 \\ \hline H_1 & H_0 \end{array} \right]$$

We may assume that no column of H_1 is a zero column because any such columns can be obtained from Y_1 . Therefore, by Lemma A.22, H_1 is a restriction of a matrix of the form $\begin{bmatrix} 1 \cdots 1 \\ I \end{bmatrix}$. By a similar argument as was used in Case 1, we see that every matroid virtually conforming to Φ is even-cycle with a blocking pair. Therefore, $\Phi \preceq \Phi_{Y_1}$, and we may discard Φ .

Now, suppose (2) of 3.5 holds. Then P_0 is of the following form, where each column of $H_{1,1}$ and $H_{0,0}$ has at most two nonzero entries and where each column of $H_{0,1}$ and $H_{1,0}$ is a unit column or a zero

column:

$$\left[\begin{array}{c|c|c|c} 1 \cdots 1 & 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 \\ 1 \cdots 1 & 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \\ \hline H_{1,1} & H_{1,0} & H_{0,1} & H_{0,0} \end{array} \right]$$

By Lemma A.22, $H_{1,1}$ is a restriction of a matrix of the form $\begin{bmatrix} 1 \cdots 1 \\ I \end{bmatrix}$, with possibly a zero column as well. Therefore, by adding the first row to the second, we see that any matroid virtually conforming to Φ is even-cycle with a blocking pair by taking the first row to be the sign row and taking the blocking pair to be represented by the second and third rows. Therefore, $\Phi \preceq \Phi_{Y_1}$, and we may discard Φ . This completes the proof. \square

4. EVEN-CUT MATROIDS

In her thesis [6], Pivotto gives several descriptions of even-cut matroids, any of which can serve as a definition. Recall that we use the following definition: An *even-cut matroid* is a matroid M that can be represented by a binary matrix with a row whose removal results in a matrix representing a cographic matroid. Thus, there is some binary extension N of M on ground set $E(M) \cup \{e\}$ such that N/e is cographic. Thus, to check if a binary matroid M is even-cut, it suffices to check if N/e is cographic for any binary extension N of M . It will be useful to consider the dual situation. Therefore, it suffices to check if there is a binary coextension N^* of M^* such that $N^* \setminus e$ is graphic. If this is the case, then $M^* \in \mathcal{M}(\Phi_C)$. We see then that $\mathcal{M}^*(\Phi_C)$ is exactly the class of even-cut matroids. Recall from Lemma 2.12 that $\Phi_{Y_1} \preceq \Phi_C$. This property reflects the fact, first observed by Pivotto [6], that even-cycle matroids with a blocking pair are duals of even-cut matroids.

In this section, we prove Theorem 1.3. Again, the computational parts of our proof were carried out in SageMath. Details, including Lemmas B.1-B.10, can be found in Appendix B.

The matroid H_{12} is the vector matroid of the binary matrix below. In that matrix, the top row is the sign row.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Lemma 4.1. *The matroids $M(K_6)$ and H_{12}^* are excluded minors for the class of even-cut matroids.*

Proof. This was verified using SageMath in Lemmas B.1 and B.2. \square

Lemma 4.1 shows that the class of even-cut matroids is contained in $EX(M(K_6), H_{12}^*)$. Theorem 1.3 shows that for sufficiently highly connected matroids, the reverse inclusion holds. A matroid M is *cyclically k -connected* if, for each partition (X, Y) of the ground set of M with $r(X) + r(Y) - r(M) < k - 1$, either X or Y is independent. Equivalently, a matroid is cyclically k -connected if its dual is vertically k -connected.

We will prove Theorem 1.3 after giving a definition and proving some lemmas. Throughout this section, recall that all templates are assumed to be in Φ -standard form (see Figure 4).

Definition 4.2. Let $|C| = 2$ and let Δ be the subgroup of \mathbb{F}^C generated by $[1, 0]$ and $[0, 1]$. The binary frame template Φ_C^2 is given by

$$\Phi_C^2 = (\{1\}, C, \emptyset, \emptyset, \emptyset, \emptyset, [\emptyset], \Delta, \{0\}).$$

Lemma 4.3. *If $\Phi = (\{1\}, C, D, \emptyset, Y_0, Y_1, A_1, \Delta, \Lambda)$ is a binary frame template, then either $\Phi_C^2 \preceq \Phi$ or Φ is equivalent to a template with $|C_1| \leq 1$.*

Proof. There are three cases to consider.

Case 1: Every element of $\Delta|C$ is in the row space of $A_1[D, C]$. Then contraction of C_0 turns the elements of C_1 into loops, and contraction of C_1 is the same as deletion of C_1 . By deleting C_1 from every matrix virtually conforming to Φ , we see that Φ is equivalent to a template with $C_1 = \emptyset$.

Case 2: There is exactly one element $\bar{x} \in \Delta|C$ that is not in the row space of $A_1[D, C]$. Then contraction of C_0 turns the elements of C_1 into parallel elements. Thus, contraction of any element $c \in C_1$ turns the elements of $C_1 - \{c\}$ into loops, and contraction of $C_1 - \{c\}$ is the same as deletion of $C_1 - \{c\}$. By deleting $C_1 - \{c\}$ from every matrix virtually conforming to Φ , we see that Φ is equivalent to a template with $|C_1| = 1$.

Case 3: There are distinct elements \bar{x} and \bar{y} in $\Delta|C$ that are not in the row space of $A_1[D, C]$. Index the elements of C_0 by $\{1, 2, \dots, n\}$ and the elements of D_0 by $\{d_1, d_2, \dots, d_n\}$. Let S_x and S_y be the supports of $\bar{x}|C_0$ and $\bar{y}|C_0$, respectively. Then the support of $(\bar{x} + \bar{y})|C_0$ is the symmetric difference $S_x \Delta S_y$. First, suppose that for every pair of elements \bar{x} and \bar{y} in $\Delta|C$ that are not in the row space of $A_1[D, C]$, we have that $\bar{x} + \bar{y}$ is in the row space of $A_1[D, C]$. Then the zero vector

is equal to

$$\sum_{i \in S_x \triangle S_y} A_1[\{d_i\}, C] + \bar{x} + \bar{y} = \sum_{i \in S_x} A_1[\{d_i\}, C] + \bar{x} + \sum_{i \in S_y} A_1[\{d_i\}, C] + \bar{y}$$

and therefore, since we are working in characteristic 2,

$$\sum_{i \in S_x} A_1[\{d_i\}, C] + \bar{x} = \sum_{i \in S_y} A_1[\{d_i\}, C] + \bar{y}.$$

Thus, contraction of C_0 projects \bar{x} and \bar{y} into the same element of \mathbb{F}^{C_1} . Moreover, this is true for any pair of elements of $\Delta|C$ that are not in the row space of $A_1[D, C]$. Therefore, the same argument used for Case 2 shows that Φ is equivalent to a template with $|C_1| = 1$.

Therefore, we may assume that there are elements \bar{x} and \bar{y} in $\Delta|C$ that are not in the row space of $A_1[D, C]$ and such that $\bar{x} + \bar{y}$ is also not in the row space of $A_1[D, C]$. Repeatedly perform operations (4) and (11) on Φ until the following template is obtained:

$$(\{1\}, C, D, \emptyset, \emptyset, \emptyset, A_1[D, C], \Delta|C, \Lambda).$$

On this template, perform operations (2) and (3) to obtain the following template:

$$(\{1\}, C, D, \emptyset, \emptyset, \emptyset, A_1[D, C], \langle \bar{x}, \bar{y} \rangle, \{0\}).$$

By performing elementary row operations, we see that every matrix virtually respecting this template is row equivalent to a matrix virtually respecting the following template, where $\bar{x}'|C_0$ and $\bar{y}'|C_0$ are zero vectors:

$$\Phi' = (\{1\}, C, D, \emptyset, \emptyset, \emptyset, A_1[D, C], \langle \bar{x}', \bar{y}' \rangle, \{0\}).$$

Note that $\bar{x}'|C_1$, $\bar{y}'|C_1$, and $(\bar{x}' + \bar{y}')|C_1$ are nonzero since \bar{x} , \bar{y} , and $\bar{x} + \bar{y}$ were not in the row space of $A_1[D, C]$ in the original template Φ . Also, we must have $\bar{x}' \neq \bar{y}'$ because otherwise, $\bar{x}' + \bar{y}' = \mathbf{0}$, contradicting the assumption that $\bar{x} + \bar{y}$ was not in the row space of $A_1[D, C]$ in Φ . Now, on Φ' , repeatedly perform operation (7) and then operation (6) to obtain the following template:

$$\Phi'' = (\{1\}, C_1, \emptyset, \emptyset, \emptyset, \emptyset, [\emptyset], \langle \bar{x}'|C_1, \bar{y}'|C_1 \rangle, \{0\}).$$

Now, any matroid M conforming to Φ'' is obtained by contracting C_1 from $M[A]$, where A is a matrix conforming to Φ'' . Thus, if there are any elements of C_1 that are parallel elements in $M[A]$, contracting one of these elements turns the rest of the parallel class into loops. So these elements are deleted to obtain M . Thus, Φ'' is equivalent to a template where these elements have been deleted from C . There are two cases to consider. First, if it is the case that either the supports of \bar{x}' and \bar{y}' are disjoint or that one support is contained in the other, then in the

resulting template, $|C| = 2$ and $\Delta = \langle [1, 0], [0, 1] \rangle$. So this resulting template is Φ_C^2 . In the other case, \bar{x}' and \bar{y}' have intersecting supports but neither is contained in the other. In this case, Φ'' is equivalent to the following template with $|C_1| = 3$:

$$\Phi''' = (\{1\}, C_1, \emptyset, \emptyset, \emptyset, \emptyset, [\emptyset], \langle [1, 1, 0], [1, 0, 1] \rangle, \{0\}).$$

However, by contracting any element of C , the other two become parallel. Thus, by contracting a second element, the third becomes a loop. Therefore, the third element is deleted to obtain a matroid conforming to Φ''' . Thus, $\Phi''' \sim \Phi_C^2$.

Similarly, the converse is true that any matroid conforming to Φ_C^2 conforms to Φ'' . Thus, $\Phi_C^2 \sim \Phi'' \preceq \Phi$. This completes the proof of the lemma. \square

Lemma 4.4. *If $\Phi = (\{1\}, C, D, \emptyset, Y_0, Y_1, A_1, \Delta, \Lambda)$ is a binary frame template with $|C_1| = 1$ and with $\Lambda|D_1$ trivial, then $\Phi_{CD} \preceq \Phi$ or Φ is equivalent to a template with $C = \emptyset$.*

Proof. We consider two cases, depending on whether $\Delta|C$ contains any element that is not in the row space of $A_1[D, C]$.

Case 1: Every element of $\Delta|C$ is in the row space of $A_1[D, C]$. Let A be a matrix that conforms to Φ . When C_0 is contracted from $M[A]$, each element of C_1 becomes a loop and can therefore be deleted rather than contracted. Thus, Φ is equivalent to a template Φ' with $|C_1| = 0$. Suppose there exist elements $\bar{x} \in \Delta|C_0$ and $\bar{y} \in \Lambda|D_0$ such that there are exactly an odd number of natural numbers i with $\bar{x}_i = \bar{y}_i = 1$. Perform operations (4), (11), and (6) on Φ' to obtain the following template:

$$(\{1\}, C_0, D_0, \emptyset, \emptyset, \emptyset, A_1[D_0, C_0], \Delta|C_0, \Lambda).$$

Then perform operations (2) and (3) to obtain the following template:

$$(\{1\}, C_0, D_0, \emptyset, \emptyset, \emptyset, A_1[D_0, C_0], \{\mathbf{0}, \bar{x}\}, \{\mathbf{0}, \bar{y}\}).$$

Any matroid conforming to this template is obtained by contracting C_0 . If \bar{x} is in the row labeled by r and \bar{y} is in the column labeled by c , then when C_0 is contracted, 1 is added to the entry of the Γ -frame matrix in row r and column c . Otherwise, the entry remains unchanged when C is contracted. We see then that this template is equivalent to Φ_{CD} , where 1s are used to replace \bar{x} and \bar{y} .

Thus, we may assume that for every element $\bar{x} \in \Delta|C_0$ and $\bar{y} \in \Lambda|D_0$, there are exactly an even number of natural numbers i such that $\bar{x}_i = \bar{y}_i = 1$. This implies that contraction of C has no effect on the Γ -frame matrix. So Φ' , and therefore Φ are equivalent to a template

with $\Lambda|D_0$ trivial. In this case, we see that repeated use of operation (7) produces a template equivalent to Φ with $C = \emptyset$.

Case 2: There is an element $\delta \in \Delta|C$ that is not in the row space of $A_1[D, C]$. Since $|C_1| = 1$, any element of $\Delta|C$ not in the row space of $A_1[D, C]$ becomes a 1 after C_0 is contracted, and any element that is in the row space becomes a 0. Therefore, we may assume that the column vector $A_1[D, C_1]$ is a zero vector and that any element of $\Delta|C$ has a 0 as its final entry if it is in the row space and a 1 otherwise.

If $\bar{x} \in \Delta|C$ and $\bar{y} \in \Lambda|D_0$ are such that there are exactly an odd number of natural numbers i such that $\bar{x}_i = \bar{y}_i = 1$, then we call the ordered pair (\bar{x}, \bar{y}) a *pair of odd type*. Otherwise, (\bar{x}, \bar{y}) is a *pair of even type*. Consider any matrix A virtually conforming to Φ and contract C from $M[A]$. The effect on the elements of Δ is a change of basis followed by a projection into a lower dimension. Therefore, a group structure is maintained. Let us call the resulting group Δ' .

Suppose there exists a pair (\bar{x}, \bar{y}) of odd type. If \bar{x} is in the row space of $A_1[D, C]$, or if \bar{x} is not in the row space of $A_1[D, C]$ but (δ, \bar{y}) is a pair of even type, then we will show that $\Phi_{CD} \preceq \Phi$. On Φ , perform operations (4), (11), and (6) to obtain the following template:

$$(\{1\}, C, D_0, \emptyset, \emptyset, \emptyset, A_1[D_0, C], \Delta|C, \Lambda).$$

Then perform operations (2) and (3) to obtain the following template:

$$\Phi' = (\{1\}, C, D_0, \emptyset, \emptyset, \emptyset, A_1[D_0, C], \langle \bar{x}, \delta \rangle, \{\mathbf{0}, \bar{y}\}).$$

Consider the following matrix conforming to Φ_{CD} :

$0 \dots 0$	$1 \dots 1$	1
		1
		\vdots
Γ -frame	Γ -frame	1
matrix	matrix	0
		\vdots
		0

The matrix below conforms to Φ' and results in the same matroid when C is contracted:

		C_0	C_1
0	$\bar{y} \cdots \bar{y}$	I	0 \vdots 0
$0 \cdots 0$	$0 \cdots 0$	δ	
Γ -frame matrix	Γ -frame matrix	\bar{x} \vdots \bar{x}	
		0	

Therefore, we may assume that an element $\delta' \in \Delta|C$ is in the row space of $A_1[D, C]$ if and only if (δ', \bar{y}) is a pair of even type. Moreover, this is true for any nonzero element of $\Lambda|D_0$. Thus, if \bar{y}_1 and \bar{y}_2 are nonzero elements of $\Lambda|D_0$, then both (δ, \bar{y}_1) and (δ, \bar{y}_2) are pairs of odd type. This implies that $(\delta, \bar{y}_1 + \bar{y}_2)$ is a pair of even type. Thus $\bar{y}_1 = \bar{y}_2$ and $\Lambda|D_0 = \{\mathbf{0}, \bar{y}\}$ in the original template Φ . By a similar argument, $\bar{x} = \delta$ and $\Delta|C = \{\mathbf{0}, \delta\}$ in the original template Φ . Therefore, Φ is equivalent to a template with $|C_0| = |D_0| = 1$, with $A_1[D_0, C_0] = [1]$, with $A_1[D_0, C_1] = [0]$, with $\Lambda|D_0 \cong GF(2)$, and with $\Delta|C$ generated by $[1, 1]$. We now assume that Φ is of this form and will show that it is equivalent to the following template Φ' , with $C = \emptyset$, obtained by adjoining an element y to Y_1 and letting $A_1[D_1, y]$ be the zero vector. We will define Δ'' below.

$$\Phi' = (\{1\}, \emptyset, D_1, \emptyset, Y_0, Y_1 \cup y, A_1[D_1, Y_0 \cup Y_1 \cup y], \Delta'', \{0\})$$

Recall that Δ' is the group obtained from Δ after C is contracted. Let Δ'' be the subgroup of $GF(2)^{Y_0 \cup Y_1 \cup y}$ consisting of all the row vectors obtained by adjoining to any element of Δ' either a zero or a 1. So $|\Delta''| = 2|\Delta'|$. Let A be a matrix that virtually conforms to Φ . Recall that columns indexed by elements of Z are formed by adding a column indexed by Y_1 to a column indexed by Z in a matrix that respects Φ . If $A[B - D, C]$ is the zero matrix, then $M[A]/C$ conforms to Φ' because we may simply choose never to use y to build a column indexed by Z . Otherwise, choose an element r of $B - D$ such that $A[r, C] = [1, 1]$. Let S be the subset of $B - (D \cup r)$ such that $s \in S$ if and only if $A[s, C] = [1, 1]$. Let $T = B - (D \cup S \cup r)$. The effect on the Γ -frame matrix of contracting C from $M[A]$ is to remove r and to add a 1 to each entry $A_{s,c}$ of the Γ -frame matrix where $s \in S$ and where c is an element of $E - (B \cup C \cup Y_0 \cup Y_1 \cup Z)$ with $A_{r,c} = 1$. Thus, in the

resulting matrix \hat{A} , for every such column c , either $\hat{A}[S, c] = [1, \dots, 1]^T$ and $\hat{A}[T, c] = [0, \dots, 0]^T$, or $\hat{A}[S, c] = [1, \dots, 1]^T$ and $\hat{A}[T, c]$ is an identity column, or $\hat{A}[S, c]$ is the complement of an identity column and $\hat{A}[T, c] = [0, \dots, 0]^T$. This exact same situation can be obtained with Φ' using the new column y . Thus, Φ is equivalent to Φ' , a template with $C = \emptyset$.

Therefore, we may assume that every pair of elements $(\bar{x}, \bar{y}) \in \Delta \times (\Lambda|D_0)$ is a pair of even type. Thus, contraction of C_0 has no effect on the Γ -frame matrix. This implies that Φ is equivalent to a template with Λ trivial. By repeated use of operation (7), we obtain a template equivalent to Φ with $C_0 = \emptyset$, with $|C_1| = 1$, and with $\Delta|C$ isomorphic to $GF(2)$. Using an argument similar to the one used above, we see that Φ is equivalent to a template with $C = \emptyset$ by adjoining an element to Y_1 . \square

Lemma 4.5. *Let $\Phi = (\{1\}, C, D, \emptyset, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a binary frame template. If $\Phi' = (\{1\}, C', D, \emptyset, \emptyset, \emptyset, A_1, \Delta, \Lambda)$, where $C' = Y_0 \cup Y_1 \cup C$, then every matroid conforming to Φ' is a minor of a matroid conforming to Φ .*

Proof. Let $\Phi'' = (\{1\}, C, D, \emptyset, Y'_0, \emptyset, A_1, \Delta, \Lambda)$, where $Y'_0 = Y_0 \cup Y_1$. By Lemma 2.13, $\Phi'' \preceq \Phi$. Any matroid conforming to Φ' is obtained from a matroid conforming to Φ'' by contracting Y'_0 . \square

We now prove Theorem 1.3, which we restate below.

Theorem 4.6. *There exist $k, l \in \mathbb{Z}_+$ such that a cyclically k -connected binary matroid with at least l elements is in $EX(M(K_6), H_{12}^*)$ if and only if it is an even-cut matroid.*

Proof. Let \mathcal{M} denote the class of binary matroids contained in $EX(M(K_6), H_{12}^*)$ and let $\mathcal{T}_{\mathcal{M}} = \{\Phi_1, \dots, \Phi_s, \Psi_1, \dots, \Psi_t\}$. Consider any template $\Phi \in \{\Phi_1, \dots, \Phi_s\}$. Recall that every matroid conforming to Φ must be contained in the minor-closed class \mathcal{M} . Every graphic matroid is a minor of a matroid that conforms to Φ . Since \mathcal{M} does not contain $M(K_6)$, it must be the case that Φ does not exist. Thus, $s = 0$ and $\mathcal{T}_{\mathcal{M}} = \{\Psi_1, \dots, \Psi_t\}$. We will study cyclically connected matroids in \mathcal{M} by considering their vertically connected duals. Thus, we will study matroids virtually conforming to some template $\Psi \in \{\Psi_1, \dots, \Psi_t\}$. Because $M(K_6)$, and H_{12}^* are cosimple matroids, it suffices to consider the simple matroids conforming to these templates.

Let $\Psi = (\{1\}, C, D, \emptyset, Y_0, Y_1, A_1, \Delta, \Lambda)$ be a template in $\mathcal{T}_{\mathcal{M}}$. We know that $\mathcal{M}^*(\Phi_C)$ is the class of even-cut matroids. Therefore, we may assume that $\Phi_C \in \{\Psi_1, \dots, \Psi_t\}$, and if any template $\Psi \preceq \Phi_C$, we

may discard Ψ from the set $\{\Psi_1, \dots, \Psi_t\}$. Since H_{12} is an even-cycle matroid, H_{12} conforms to Φ_D . Thus, we have $\Phi_D \not\leq \Psi$. By Lemma 2.14, $\Lambda|D_1$ is trivial. Moreover, by Lemma 2.12, we have $\Phi_{CD} \not\leq \Psi$.

The following matrix conforms to Φ_C^2 , with C indexing the last two columns:

$$\left[\begin{array}{cccccccccccc|cc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right].$$

By contracting C , we obtain the following matrix A with $M[A]$ conforming to Φ_C^2 :

$$A = \left[\begin{array}{cccccccccccc} 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right].$$

By adding the first and third rows to the fifth row, we see that this matrix represents H_{12} . Therefore, $\Phi_C^2 \not\leq \Psi$ and by Lemma 4.3, we may assume that $|C_1| \leq 1$.

Since $\Phi_{CD} \not\leq \Psi$, Lemma 4.4 implies that Ψ is equivalent to a template with $C = \emptyset$. Hence we will assume from now on that $C = \emptyset$.

Since $C_0 = \emptyset$, we have $D_0 = \emptyset$. Also, we have seen that $\Lambda|D_1$ is trivial. Therefore, Λ is trivial. By performing elementary row operations on every matrix respecting Ψ , we may assume that A_1 is of the following form, with $Y_0 = V_0 \cup V_1$, with $Y_1 = Y'_1 \cup Y''_1$, and with the stars representing arbitrary binary matrices:

$$\begin{array}{|c|c|c|c|} \hline Y'_1 & Y''_1 & V_0 & V_1 \\ \hline I & * & 0 & Q \\ \hline 0 & 0 & I & * \\ \hline \end{array}.$$

Also, by elementary row operations, we may assume that $\Delta|(Y'_1 \cup V_0)$ is trivial.

We will now show that $|\Delta| \leq 2$. Suppose otherwise. Then Δ contains a subgroup Δ' isomorphic to $GF(2) \times GF(2)$. Repeatedly perform y -shifts and operation (13) and then perform operation (3) to obtain the following template:

$$(\{1\}, \emptyset, \emptyset, \emptyset, Y_1'' \cup V_1, \emptyset, [\emptyset], \Delta', \{0\}).$$

By Lemma 4.5, if $C' = Y_1'' \cup V_1$, then every matroid conforming to the following template is a minor of a matroid conforming to Ψ :

$$(\{1\}, C', \emptyset, \emptyset, \emptyset, \emptyset, [\emptyset], \Delta', \{0\})$$

The latter template is equivalent to Φ_C^2 since $\Delta' \cong \langle [1, 0], [0, 1] \rangle$. Thus, $|\Delta| \leq 2$. Therefore, there is at most one nonzero element $\bar{x} \in \Delta$. Let $Y_{1,i}$ consist of the elements $y \in Y_1''$ such that $\bar{x}_y = i$. Similarly, let $Y_{0,i}$ consist of the elements $y \in V_1$ such that $\bar{x}_y = i$.

Let M be a maximum sized simple matroid of rank $r(M)$ virtually conforming to Ψ . So M uses all possible columns indexed by Z and has a Γ -frame matrix with n rows representing the complete graph K_{n+1} . Let r and λ be the rank and connectivity functions of M respectively, and let r_Q be the rank of $M[Q]$. Then $r(Y_0) \leq |V_0| + r_Q + 1$, where the 1 comes from the group Δ . We have $r(E(M) - Y_0) = |Y_1'| + n$ and $r(M) = |Y_1'| + |V_0| + n$. Thus, $\lambda(Y_0) \leq r_Q + 1$. Let N be any matroid virtually conforming to Ψ . Then $\lambda_N(Y_0) \leq \lambda(Y_0) \leq r_Q + 1$. Thus, if we set $k > r_Q + 2$, then no simple vertically k -connected matroid N virtually conforms to Ψ unless Y_0 or $E(N) - Y_0$ is spanning. The set Y_0 can only be spanning if the Γ -frame matrix has at most one row. In that case, N can only be simple if $|E(N)| \leq 2|Y_1| + |Y_0| + 1$. If we set l greater than this value, then no simple, vertically k -connected matroid N with at least l elements conforms to Ψ unless $E(N) - Y_0$ is spanning, that is unless $V_0 = \emptyset$. Thus, A_1 is of the following form, where $Y_1 = Y_1' \cup Y_{1,0} \cup Y_{1,1}$ and where $Y_0 = Y_{0,0} \cup Y_{0,1}$:

$$\begin{array}{c|c|c|c|c} Y_1' & Y_{1,0} & Y_{1,1} & Y_{0,0} & Y_{0,1} \\ \hline I & A_{Y_1} & B_{Y_1} & A_{Y_0} & B_{Y_0} \end{array}.$$

By Lemma B.3, each row of B_{Y_1} consists either entirely of 0s or entirely of 1s. Any duplicate columns in either $[I|A_{Y_1}]$ or B_{Y_1} produce the same columns in a matrix virtually conforming to Ψ . Therefore, we may assume that $|Y_{1,1}| \leq 1$, that every column of A_{Y_1} contains at least two nonzero entries, and that no column of A_{Y_1} is a copy of another. Since we are only considering templates to which simple matroids conform, we may assume that no column of A_{Y_0} is a copy of another and also that no column of B_{Y_0} is a copy of another. By Lemma B.4, either $Y_{1,0}$ or $Y_{1,1}$ is empty. If $|Y_{1,0}| \geq 2$, then A_{Y_1} contains one of the submatrices below, all of which are forbidden by Lemma

B.5. Therefore, $|Y_{1,0}| \leq 1$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Claim 4.6.1 *The following matrices are forbidden if the column on the left is indexed by an element of Y_1 and the column on the right is indexed by an element of Y_0 :*

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Proof of Claim 4.6.1. If the column on the left is contained in A_{Y_1} and the column on the right is contained in A_{Y_0} , then these matrices are forbidden because contraction of the element indexing this column of A_{Y_0} produces a new A_{Y_1} , containing a column originally in the identity matrix, that contains a submatrix forbidden by Lemma B.5. Since we may choose the zero vector for every element of Δ , these submatrices are also forbidden from $[A_{Y_1}|B_{Y_0}]$, from $[B_{Y_1}|B_{Y_0}]$, and from $[B_{Y_1}|A_{Y_0}]$. This completes the proof of the claim.

By Lemma B.6, if $|Y_{1,0}| = 1$, then $Y_{0,1} = \emptyset$. By Lemma B.7, if $|Y_{1,1}| = 1$, then each column of A_{Y_0} contains at most two nonzero entries.

Recall that a binary matroid M conforms to Φ_C if there is some binary coextension N of M on ground set $E(M) \cup \{e\}$ such that $N \setminus e$ is graphic. Thus, checking if a binary matroid conforms to Φ_C amounts to checking if some row can be added to the matrix to make the resulting matroid graphic. There are four cases to check:

Case I: $|Y_{1,0}| = 1$

Case II: $|Y_{1,1}| = 1$

Case III: $Y_{1,0} = Y_{1,1} = Y_{0,1} = \emptyset$

Case IV: $Y_{1,0} = Y_{1,1} = \emptyset$ and $Y_{0,1} \neq \emptyset$

In the diagrams of matrices below, we will use the abbreviations *n.p.c.* and *z.p.c.* to stand for “nonzero entries per column” and “zeros per column” respectively.

Case I: Since $|Y_{1,0}| = 1$, the arguments above imply that $Y_{1,1} = Y_{0,1} = \emptyset$. We study the matrix A_{Y_0} . Let $D = R \cup S$, where R is the set of rows where A_{Y_1} has its nonzero entries. We will show that $\Psi \preceq \Phi_C$

by adding to every matrix virtually conforming to Ψ a row indexed by d so that the resulting matroid is graphic. The row indexed by d also has a 1 in the column indexed by $Y_{1,0}$, and we are adding this row to every row in R . The form of A_{Y_0} itself (without the row indexed by d) is determined by Claim 4.6.1.

First, we study A_{Y_0} when $|R| = 2$.

d	0	0	$1 \cdots 1$
R	0	1 n.p.c.	2 n.p.c.
S	≤ 2 n.p.c.	≤ 1 n.p.c.	≤ 1 n.p.c.

Next, we study A_{Y_0} when $|R| = 3$.

d	0	0	0	$1 \cdots 1$
R	0	1 n.p.c.	2 n.p.c.	3 n.p.c.
S	≤ 2 n.p.c.	≤ 1 n.p.c.	0	≤ 1 n.p.c.

Next, we study A_{Y_0} when $|R| \geq 4$. Here, J denotes a matrix where every entry is a 1.

d	0	0	0	$1 \cdots 1$	$1 \cdots 1$
R	0	1 n.p.c.	2 n.p.c.	1 z.p.c.	J
S	≤ 2 n.p.c.	≤ 1 n.p.c.	0	0	≤ 1 n.p.c.

Case II: Since $|Y_{1,1}| = 1$, the arguments above implies that $|Y_{1,0}| = \emptyset$ and that each column of A_{Y_0} has at most two nonzero entries. We study the matrix B_{Y_0} . By Lemma B.8, the submatrices below, with the column to the left of the vertical line contained in B_{Y_1} , and the column to the right of the vertical line contained in B_{Y_0} , are forbidden.

$$\left[\begin{array}{c|c} 0 & 1 \\ 0 & 1 \end{array} \right], \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{c|c} 1 & 0 \\ 1 & 0 \end{array} \right]$$

This fact, along with Claim 4.6.1, determines the form of B_{Y_0} .

Let $D = R \cup S$, where R is the set of rows where B_{Y_1} has its nonzero entries. We will show that $\Psi \preceq \Phi_C$ by adding to every matrix virtually conforming to Ψ a row indexed by d so that the resulting matroid is graphic. The row indexed by d also has a 1 in the column indexed by $Y_{1,0}$, and we are adding this row to every row in R , as well as to every row indexed by an element of $B - D$ where the nonzero element \bar{x} of Δ is used.

First, we study the case when $R = \emptyset$.

	$Y_{0,1}$	$Y_{0,0}$
d	$1 \cdots 1$	$0 \cdots 0$
S	≤ 1 n.p.c.	$A_{Y_0} (\leq 2 \text{ n.p.c.})$

Now we study the case when $R \neq \emptyset$. Here J denotes a matrix where every entry is a 1.

	$Y_{0,1}$	$Y_{0,0}$
d	$1 \cdots 1$	$0 \cdots 0$
R	1 z.p.c.	A_{Y_0}
S	0	$(\leq 2 \text{ n.p.c.})$

Case III: Since $Y_{1,0} = Y_{1,1} = Y_{0,1} = \emptyset$, we have $Y_1 = Y'_1$ and $Y_0 = Y_{0,0}$. The submatrices below are forbidden from A_{Y_0} because by deleting the rest of Y_0 and contracting the elements of Y_0 indexing the two given columns, we produce one of the submatrices forbidden by Lemma B.5. Since we may choose the zero vector for every element of Δ , these submatrices are also forbidden from B_{Y_0} .

$$Q_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad Q_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad Q_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad Q_4 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

If every column of A_{Y_0} has at most two nonzero entries, then $\Psi \sim \Phi_0 \preceq \Phi_C$ and can be discarded. Thus, we may assume that there is a column of A_{Y_0} with at least three nonzero entries. Let H be the submatrix of A_{Y_0} consisting of all the columns with at most two nonzero entries.

Let y be an element of Y_0 such that $A_{Y_0}[D, \{y\}]$ has a maximum number of nonzero entries among all elements of Y_0 . Let $D = R \cup S$, where $(A_{Y_0})_{r,y} = 1$ for each $r \in R$ and $(A_{Y_0})_{s,y} = 0$ for each $s \in S$. We will prove the following.

Claim 4.6.2 *Let v be a column of A_{Y_0} such that $v|R$ has at least two zeros. Then either v has at most two nonzero entries, or $v|R$ has exactly two zeros and $v|S$ is a zero vector.*

Proof of Claim 4.6.2. If there is a column v of A_{Y_0} such that $v|R$ has exactly two zeros, then if $|R| = 3$, the fact that Q_2 is forbidden implies that v has at most two nonzero entries. If $|R| > 3$, then the fact that Q_1 is forbidden implies that $v|S$ is a zero vector. Now, if there is a column v of A_{Y_0} such that $v|R$ has at least three zeros, then since Q_4 is forbidden, $v|R$ has at most two nonzero entries. Since Q_1 , Q_2 , and Q_3 are forbidden, v has at most two nonzero entries. This completes the proof of the claim.

Suppose there are two elements other than y that index columns v_1 and v_2 of A_{Y_0} with $|R|$ nonzero entries. Since Q_2 is forbidden, $v_1|R$ and $v_2|R$ each have at most one zero. Since we are only considering simple matroids conforming to Ψ , we have that $v_1|R$ and $v_2|R$ each have exactly one zero. If $v_1|R$ and $v_2|R$ have their zeros in different rows, then again since Q_2 is forbidden, the nonzero entries in $v_1|S$ and $v_2|S$ must be in the same row. Thus, we may divide this case into three subcases:

- (1) There is at least one column other than the one indexed by y with $|R|$ nonzero entries, and all such columns have a zero in the same row r of R .
- (2) There are at least two columns other than the one indexed by y with $|R|$ nonzero entries, and all such columns have a nonzero entry in the same row s of S .
- (3) No column other than the one indexed by y has $|R|$ nonzero entries.

In subcase (1), by Claim 4.6.2, if v is a column of A_{Y_0} such that $v|R$ has two zeros, either v is a column of H or $v|R$ has exactly two zeros and $v|S$ is a zero vector. If v is not a column of H but is such that $v|R$ has exactly two zeros, then $|R| \geq 5$. Since Q_1 is forbidden, v has a zero in the row indexed by r . Therefore, $A_{Y_0}[D, Y_0 - y]$ has the following form, where J denotes a matrix where every entry is a 1:

$$\begin{array}{c} R - r \\ r \\ S \end{array} \begin{array}{|c|c|c|} \hline J & 1 \text{ z.p.c.} & \\ \hline 0 \cdots 0 & 0 \cdots 0 & \\ \hline \leq 1 \text{ n.p.c.} & 0 & \\ \hline \end{array} \begin{array}{c} \\ H \\ \end{array}$$

Below, we add the row d to the matrix, where d also has a 1 in the entry in the column of y . By adding d to every row in $R - r$, we see that the resulting matroid is graphic. Thus, in this subcase, $\Psi \preceq \Phi_C$.

$$\begin{array}{c} d \\ R - r \\ r \\ S \end{array} \begin{array}{|c|c|c|} \hline 1 \cdots 1 & 1 \cdots 1 & 0 \cdots 0 \\ \hline J & 1 \text{ z.p.c.} & \\ \hline 0 \cdots 0 & 0 \cdots 0 & \\ \hline \leq 1 \text{ n.p.c.} & 0 & \\ \hline \end{array} \begin{array}{c} \\ H \\ \end{array}$$

We now consider subcase (2). Suppose there is some column v of A_{Y_0} such that $v|R$ has two zeros. If v has more than two nonzero entries, then Claim 4.6.2 implies that $v|S$ is a zero vector. By Lemma B.9, along with the facts that Q_1 is forbidden and that we are only considering templates to which simple matroids conform, no such column v can exist. Therefore, $A_{Y_0}[D, Y_0 - y]$ has the following form:

R	1 z.p.c.	H
s	$1 \cdots 1$	
$S - s$	0	

Below, we add the row d to the matrix, where d also has a 1 in the entry in the column of y . By adding d to every row in $R \cup \{s\}$, we see that the resulting matroid is graphic. Thus, in this subcase, $\Psi \preceq \Phi_C$.

d	$1 \cdots 1$	$0 \cdots 0$	H
R	1 z.p.c.		
s	$1 \cdots 1$		
$S - s$	0		

We now consider subcase (3). If $|R| = 3$, then the assumption that no column other than the one indexed by y has $|R|$ nonzero entries implies that $A_{Y_0}[D, Y_0 - y] = H$. Therefore, we may assume that $|R| \geq 4$. Let v be a column of A_{Y_0} such that $v|R$ has exactly one zero. Then the assumption that no column other than the one indexed by y has $|R|$ nonzero entries implies that $v|S$ is a zero vector. If w is a column of A_{Y_0} such that $w|R$ has exactly two zeros, then Claim 4.6.2 implies that $w|S$ is a zero vector. Suppose that such a column w exists and that $|R| \geq 5$ so that w has at least three nonzero entries. Since Q_1 and Q_2 are forbidden, all columns of $A_{Y_0}[D, Y_0 - y]$ not contained in H must have a zero in the same row r of R . Since we are only considering simple matroids, there is at most one column v where $v|R$ has exactly one zero. Therefore, $A_{Y_0}[D, Y_0 - y]$ has the following form, with or without v :

	v			
$R - r$	1	1 z.p.c	H	
	\vdots			
	1			
r	0	$0 \dots 0$		
S	0	0		
	\vdots			
	0			

We add the row d to the matrix, where d also has a 1 in the entry in the column of y . By adding d to every row in $R - r$, we see that the resulting matroid is graphic.

	v		
d	1	$1 \cdots 1$	$0 \cdots 0$
$R - r$	1	1 z.p.c	H
	\vdots		
1			
r	0	$0 \cdots 0$	
S	0	0	
	\vdots		
	0		

Therefore, we may assume that no column w of A_{Y_0} exists such that $w|R$ has two zeros and at least three nonzero entries. Thus, $A_{Y_0}[D, Y_0 - y]$ is of the following form:

R	1 z.p.c.	H
S	0	

We add the row d to the matrix, where d also has a 1 in the entry in the column of y . By adding d to every row in R , we see that the resulting matroid is graphic. Thus, in this subcase, $\Psi \preceq \Phi_C$.

d	$1 \cdots 1$	$0 \cdots 0$
R	1 z.p.c.	H
S	0	

Case IV: If any column of A_{Y_0} has three nonzero entries, then by contracting that element of $Y_{0,0}$ we make a column of the identity matrix into a column of A_{Y_1} with two nonzero entries. Since $Y_{0,1}$ is nonempty, this is forbidden by Lemma B.6. Therefore, each column of A_{Y_0} contains at most two nonzero entries.

The matrices Q_5 and Q_6 below are forbidden from B_{Y_0} because by contracting one of the corresponding elements of Y_0 , we obtain, using a column of the identity matrix, a submatrix forbidden by Lemma B.8. The matrix Q_7 below is forbidden by Lemma B.10.

$$Q_5 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad Q_6 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad Q_7 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Let y be an element of Y_0 such that $B_{Y_0}[D, \{y\}]$ has a maximum number of nonzero entries among all elements of $Y_{0,1}$. Let $D = R \cup S$, where $(B_{Y_0})_{r,y} = 1$ for each $r \in R$ and $(B_{Y_0})_{s,y} = 0$ for each $s \in S$. If $|R|$ is 0 or 1, then we add to each matrix A virtually conforming to Ψ a row which is the characteristic vector of $Y_{0,1}$. If we add this row to each row of A where \bar{x} has been used as the element of Δ , we see

that the resulting matroid is graphic. Therefore, we may assume that $|R| \geq 2$.

Since Q_5 is forbidden, each column of $B_{Y_0}[R, Y_{0,1}]$ contains at most two zeros. Since Q_6 is forbidden, any column w such that $w|R$ has two zeros must be such that $w|S$ is a zero vector.

Suppose there are two columns v_1, v_2 of B_{Y_0} , in addition to the column indexed by y , with $|R|$ nonzero entries. Since Q_6 is forbidden, $v_1|R$ and $v_2|R$ each have at most one zero. Since we are only considering simple matroids conforming to Ψ , $v_1|R$ and $v_2|R$ each have exactly one zero. If $v_1|R$ and $v_2|R$ have their zeros in different rows, then again since Q_6 is forbidden, the nonzero entries in $v_1|S$ and $v_2|S$ must be in the same row. Thus, we have the same three subcases as we did in Case III. In each subcase, we will determine the structure of $B_{Y_0}[D, Y_{0,1} - y]$. Since we are only considering simple matroids conforming to Ψ , we may assume that no column of B_{Y_0} is a copy of another.

Let us consider subcase (1). If there is a column of v of $B_{Y_0}[R, Y_{0,1}]$ with two zeros, then since Q_6 is forbidden one of the zeros of v must be in row r . Therefore, $B_{Y_0}[D, Y_{0,1} - y]$ is of the following form, where J denotes a matrix where every entry is a 1:

$$\begin{array}{c} R - r \\ r \\ S \end{array} \begin{array}{|c|c|} \hline 1 \text{ z.p.c.} & J \\ \hline 0 \cdots 0 & 0 \cdots 0 \\ \hline 0 & \leq 1 \text{ n.p.c.} \\ \hline \end{array}$$

Add to every matrix A conforming to Ψ an additional row that is the characteristic vector of $Y_{0,1}$. By adding this characteristic vector to each row of A where \bar{x} has been used as the element of Δ as well as to each row of $R - r$, we see that the resulting matroid is graphic. Therefore, $\Psi \preceq \Phi_C$.

Now, we consider subcase (2). Then there are columns v_1 and v_2 of B_{Y_0} , other than the column indexed by y , with $|R|$ nonzero entries. Suppose w is a column of B_{Y_0} such that $w|R$ has two zeros. Since Q_6 is forbidden, w must have a zero in each of the rows where v_1 and v_2 have their zeros. But then B_{Y_0} contains Q_7 . Therefore, no column of $B_{Y_0}[R, Y_{0,1}]$ has two zeros. Therefore, recalling that each column of $B_{Y_0}[R, Y_{0,1}]$ has at most two zeros, we have that $B_{Y_0}[D, Y_{0,1} - y]$ is of the following form:

$$\begin{array}{c} R \\ s \\ S - s \end{array} \begin{array}{|c|} \hline 1 \text{ z.p.c.} \\ \hline 1 \cdots 1 \\ \hline 0 \\ \hline \end{array}$$

Add to every matrix A conforming to Ψ an additional row that is the characteristic vector of $Y_{0,1}$. By adding this characteristic vector to

each row of A where \bar{x} has been used as the element of Δ as well as to each row of $R \cup \{s\}$, we see that the resulting matroid is graphic. Therefore, $\Psi \preceq \Phi_C$.

Now, we consider subcase (3). If $|R| = 2$, then since Q_7 is forbidden B_{Y_0} (including the column indexed by y) is a submatrix, obtained by deleting columns or any rows but the first two, of one of the following matrices :

$$T_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \quad T_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

Add to every matrix A conforming to Ψ an additional row that is the characteristic vector of $Y_{0,1}$. If B_{Y_0} is a submatrix of T_i , then add this characteristic vector to the first i rows of A as well as to every row of A where \bar{x} has been used as the element of Δ . We see that the resulting matroid is graphic. Therefore, $\Psi \preceq \Phi_C$. Thus, we may assume that $|R| > 2$.

Recall that each column of $B_{Y_0}[R, Y_{0,1}]$ has at most two zeros. Suppose there is a column w of B_{Y_0} such that $w|R$ has exactly two zeros. Since Q_6 and Q_7 are forbidden, all columns of $B_{Y_0}[D, Y_{0,1} - y]$ must have a zero in the same row r of R . Since we are only considering simple matroids, there is at most one column v where $v|R$ has exactly one zero. Therefore, $B_{Y_0}[D, Y_{0,1} - y]$ has the following form, with or without v :

	v	
	1	
$R - r$	\vdots	1 z.p.c
	1	
r	0	0 \cdots 0
	0	
S	\vdots	0
	0	

Add to every matrix A conforming to Ψ an additional row that is the characteristic vector of $Y_{0,1}$ and add this characteristic vector to every row of $R - r$ as well as to every row of A where \bar{x} has been used as the element of Δ . We see that the resulting matroid is graphic.

Therefore, we may assume that every column of $B_{Y_0}[R, Y_{0,1} - y]$ has exactly one zero. Thus, $B_{Y_0}[D, Y_{0,1} - y]$ is of the following form:

$$\begin{array}{c} R \\ S \end{array} \begin{array}{|c|} \hline 1 \text{ z.p.c.} \\ \hline 0 \\ \hline \end{array}$$

Add to every matrix A conforming to Ψ an additional row that is the characteristic vector of $Y_{0,1}$. By adding this characteristic vector to each row of A where \bar{x} has been used as the element of Δ as well as to each row of R , we see that the resulting matroid is graphic. Therefore, $\Psi \preceq \Phi_C$. This completes the proof. \square

COMPUTATIONS FOR THE PAPER “THE HIGHLY CONNECTED
EVEN-CYCLE AND EVEN-CUT MATROIDS”

These appendices contain the computations necessary to verify some of the results claimed in this paper. The computations were carried out in Version 7.3 of SageMath [7], in particular making use of the *matroids* component [5]. We used the SageMathCloud online interface. The `.sagews` source files accompany this paper on arXiv, under the Ancillary Files. PDF printouts of these computations follow.

Appendix A. Even-cycle computations

10/1/2016

Lemma A.1. *The matroid $M^*(K_6)$ is an excluded minor for the class of even-cycle matroids.*

Proof. Below, we show that for no binary extension N of $M^*(K_6)$ on ground set $E(M) \cup \{e\}$ is N/e graphic. Thus, $M^*(K_6)$ is not an even-cycle matroid. Then, we show that the unique matroid $M^*(K_6) \setminus e$ obtained from $M^*(K_6)$ by deleting an element and the unique matroid $M^*(K_6)/e$ obtained from $M^*(K_6)$ by contracting an element each have binary extensions N such that N/e is graphic. Therefore, $M^*(K_6) \setminus e$ and $M^*(K_6)/e$ are both even-cycle matroids. \square

```
M = matroids.CompleteGraphic(6).dual()
for N in M.linear_extensions('e'):
    if (N / 'e').is_graphic():
        print N.representation()
        break
```

```
M=matroids.CompleteGraphic(6).dual()\{0}
for N in M.linear_extensions('e'):
    if (N / 'e').is_graphic():
        print N.representation()
        break
```

```
[ 1  0  0  0  0  0  0  0  0  0  0 -1  0  0  0  0]
[ 0  1  0  0  0  0  0  0  0  0  0 -1  0  0 -1]
[ 0  0  1  0  0  0  0  0  0  0  0  0 -1  0  0]
[ 0  0  0  1  0  0  0  0  0  0  0  0  0 -1  1]
[ 0  0  0  0  1  0  0  0  0  0  1 -1  0  0  0]
[ 0  0  0  0  0  1  0  0  0  0  1  0 -1  0  1]
[ 0  0  0  0  0  0  1  0  0  0  1  0  0 -1  1]
[ 0  0  0  0  0  0  0  1  0  0  0  1 -1  0  1]
[ 0  0  0  0  0  0  0  0  1  0  0  1  0 -1  1]
[ 0  0  0  0  0  0  0  0  0  1  0  0  1 -1  0]
```

```
M=matroids.CompleteGraphic(6).dual()/\{0}
for N in M.linear_extensions('e'):
    if (N / 'e').is_graphic():
        print N.representation()
        break
```

```

[ 1  0  0  0  0  0  0  0  0  0  1 -1  0  0 -1 -1]
[ 0  1  0  0  0  0  0  0  0  0  1  0 -1  0 -1  0]
[ 0  0  1  0  0  0  0  0  0  0  1  0  0 -1 -1  0]
[ 0  0  0  1  0  0  0  0  0  0  1 -1  0  0  0  0]
[ 0  0  0  0  1  0  0  0  0  0  1  0 -1  0  0  0]
[ 0  0  0  0  0  1  0  0  0  0  1  0  0 -1  0  0]
[ 0  0  0  0  0  0  1  0  0  0  1 -1  0  0  0  1]
[ 0  0  0  0  0  0  0  1  0  0  1  0 -1  0  0  1]
[ 0  0  0  0  0  0  0  0  1  0  0  1 -1  0  0  0]

```

Lemma A.2. *The matroid L_{11} is an excluded minor for the class of even-cycle matroids.*

Proof. First, we show that L_{11} is not an even-cycle matroid. Then, we show that for each of the 11 elements $a \in E(L_{11})$, both $L_{11} \setminus a$ and L_{11}/a are even-cycle matroids. \square

```

A = Matrix(GF(2), [[1,0,0,0,0,0,1,0,1,0,1],[0,1,0,0,0,0,1,0,0,1,1],
                  [0,0,1,0,0,0,0,1,1,1,0],[0,0,0,1,0,0,0,1,1,0,1],
                  [0,0,0,0,1,0,0,1,0,1,1],[0,0,0,0,0,1,0,0,1,1,1]])

```

```

L11=Matroid(field=GF(2), matrix=A)
for N in L11.linear_extensions('e'):
    if (N / 'e').is_graphic():
        print N.representation()
        break

```

```

for a in L11.groundset():
    M1=L11\ a
    for N in M1.linear_extensions('e'):
        if (N / 'e').is_graphic():
            print a
            break
    M2=L11/a
    for N in M2.linear_extensions('e'):
        if (N / 'e').is_graphic():
            print a
            break

```

```

0
0
1
1
2
2
3
3
4
4

```

5
5
6
6
7
7
8
8
9
9
10
10

For the remainder of this worksheet, Φ is a binary frame template of the form

$$\Phi = (\{1\}, \emptyset, D, \emptyset, Y_0, Y_1, A_1, \{0\}, \{0\})$$

with $Y_1 = Y'_1 \cup Y''_1$, with $A_1[D, Y'_1]$ an identity matrix, with $A_1[D, Y''_1] = P_1$, and with $A_1[D, Y_0] = P_0$. The Sage function below, which we call `template_test`, builds the largest possible simple matroid M of rank $r(M[A_1]) + n - 1$ that virtually conforms to Φ . It does this by choosing for the Γ -frame matrix the matrix representation of $M(K_n)$ and by including all possible elements of Z that can be constructed from elements of Y_1 . For AY_0 , we input the matrix $A_1[D, Y_0]$. For AY_1 , we input the matrix $A_1[D, Y_1]$. We then test if M contains $PG(3, 2) \setminus e$ as a minor by looking for and printing a subset of the ground set of M whose contraction results in a rank-4 binary matroid whose simplification has size at least 14. If this subset exists, then M must contain $PG(3, 2) \setminus e$ as a minor. In the Python programming language, on which SageMath is based, a set of size n has elements labeled $0, 1, \dots, n - 1$. Thus, for example, if $\{17, 12, 7\}$ is the subset of the ground set that is to be contracted, then the 18th, 13th, and 8th columns are to be contracted.

In all of the computations below, we show that if $\mathcal{M}_w(\Phi) \subseteq EX(PG(3, 2) \setminus e, M^*(K_6), L_{11})$, then the matrix A_1 cannot contain certain submatrices. We do this by testing the template Φ' obtained from Φ by restricting A_1 to the rows and columns of the submatrix.

```
def template_test(n, AY1, AY0):
    """
    Generate the universal matroid conforming to the relevant \
    template, with a n-vertex complete graph, for a total rank of n\
    -1+k.
    """
    num_elts = n * (n-1) / 2 + n * AY1.ncols() + AY0.ncols()
    A = Matrix(GF(2), n-1 + AY1.nrows(), num_elts)
    i = 0
    k = AY1.nrows()
    # Identity in front
    for j in range(n-1):
        A[j+k, j] = 1
    i = n-1
    # All columns with two nonzero entries in the Gamma-frame matrix
```

```

for S in Subsets(range(k,n+k-1),2):
    for j in S:
        A[j,i] = 1
        i = i + 1
    # Columns from Y1 with unit columns and all-zero columns below \
    it
    for l in range(AY1.ncols()):
        for j in range(k):
            A[j,i] = AY1[j,l]
            i = i + 1
        for m in range(n-1):
            A[k+m, i] = 1
            for j in range(k):
                A[j,i] = AY1[j,l]
                i = i + 1
    # Columns from Y0
    for l in range(AY0.ncols()):
        for j in range(k):
            A[j, i] = AY0[j, l]
            i=i+1
return A

```

Lemma A.3. *If P_1 contains a column with four or more nonzero entries, then $\mathcal{M}_w(\Phi) \not\subseteq EX(PG(3,2)\backslash e)$. More generally, if $M[A_1[D, Y_1]]$ contains a circuit of size at least 5, then $\mathcal{M}_w(\Phi) \not\subseteq EX(PG(3,2)\backslash e)$.*

Proof. See the computation below. \square

```

AY0 = Matrix(GF(2), 4, 0)
AY1 = Matrix(GF(2), [[1,0,0,0,1],[0,1,0,0,1],[0,0,1,0,1], \
    [0,0,0,1,1]])
print AY0
print AY1
A=template_test(4, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break
[]
[1 0 0 0 1]
[0 1 0 0 1]
[0 0 1 0 1]
[0 0 0 1 1]
[0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 1 1 1 1]

```

```

[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0]
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1]
{17, 12, 7}

```

Lemma A.4. *If P_1 contains the submatrix*

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

then $\mathcal{M}_w(\Phi) \not\subseteq EX(PG(3,2)\backslash e)$.

Proof. See the computation below. \square

```

AY0 = Matrix(GF(2), 4, 0)
AY1 = Matrix(GF(2), \
    [[1,0,0,0,1,0],[0,1,0,0,1,0],[0,0,1,0,0,1],[0,0,0,1,0,1]])
print AY0
print AY1
A=template_test(4, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

[]
[1 0 0 0 1 0]
[0 1 0 0 1 0]
[0 0 1 0 0 1]
[0 0 0 1 0 1]
[0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0]
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1]
{25, 12, 7}

```

Lemma A.5. *If P_1 contains the submatrix*

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

then $\mathcal{M}_w(\Phi) \not\subseteq EX(PG(3,2) \setminus e)$.

Proof. See the computation below. \square

```

AY0 = Matrix(GF(2), 3, 0)
AY1 = Matrix(GF(2), [[1,0,0,1,0,1],[0,1,0,1,1,0],[0,0,1,0,1,1]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break
[]
[1 0 0 1 0 1]
[0 1 0 1 1 0]
[0 0 1 0 1 1]
[0 0 0 1 1 1 0 0 0 0 0 0 1 1 1 0 0 0 1 1 1]
[0 0 0 0 0 1 1 1 0 0 0 1 1 1 1 1 1 0 0 0]
[0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1 1 1 1 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1]
{4}

```

Lemma A.6. *If P_1 contains the submatrix*

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

then $\mathcal{M}_w(\Phi) \not\subseteq EX(PG(3,2) \setminus e)$.

Proof. See the computation below. \square

```

AY0 = Matrix(GF(2), 3, 0)
AY1 = Matrix(GF(2), [[1,0,0,1,1,1],[0,1,0,1,0,1],[0,0,1,0,1,1]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)

```

```

print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

[]
[1 0 0 1 1 1]
[0 1 0 1 0 1]
[0 0 1 0 1 1]
[0 0 0 1 1 1 0 0 0 0 0 0 1 1 1 1 1 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 1 1 1 0 0 0 1 1 1]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1 1 1 1 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1]
{4}

```

Lemma A.7. *If P_1 contains the submatrix*

$$P'_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix},$$

then $\mathcal{M}_w(\Phi) \not\subseteq EX(PG(3,2)\backslash e)$.

Proof. In the matrix $[I_4|P'_1]$, columns 1, 2, 4, 5, and 6 form a circuit of size 5, which is forbidden by Lemma A.3. \square

Lemma A.8. *If P_0 contains a column with five nonzero entries or either of the following matrices below as a submatrix, then $\mathcal{M}_w(\Phi) \not\subseteq EX(PG(3,2)\backslash e)$:*

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Proof. If we contract a column from Y_0 , then a column from the identity matrix $A_1[D, Y'_1]$ becomes a column in $A_1[D, Y''_1]$. We see then that if P_0 contains one of the submatrices listed in this result, then contraction of the columns of that submatrix results in the submatrix forbidden in Lemma A.3, A.4, or A.7.

Lemma A.9. *If P_0 contains the columns*

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},$$

then either:

- (1) $Mw(\Phi) \not\subseteq EX(PG(3,2) \setminus e, L_{11})$ and $Mw(\Phi) \not\subseteq EX(PG(3,2)_{-2})$ or
- (2) P_0 is of the following form, where each column of $[H_1|H_0]$ has at most two nonzero entries:

$$\left[\begin{array}{c|c} 1 \cdots 1 & 0 \cdots 0 \\ \hline H_1 & H_0 \end{array} \right].$$

Proof. Suppose neither (1) nor (2) holds. Since (1) does not hold, Lemma A.8 implies that P_0 contains none of the following submatrices, with x either 0 or 1, and also contains no column with five nonzero entries:

$$\begin{bmatrix} 1 & 1 & x \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & x \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & x \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, since (2) does not hold, P_0 contains one of the following submatrices:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The computations below show that each of these submatrices results in a template to which virtually conforms a matroid which has $PG(3,2) \setminus e$ as a minor or which has both $PG(3,2)_{-2}$ and L_{11} as minors, contradicting the assumption that (1) does not hold. \square

```
AY0 = Matrix(GF(2), \
    [[1,1,1],[1,0,1],[1,0,0],[0,1,1],[0,1,0],[0,0,1]])
AY1 = matrix.identity(6)
```

```

print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1 1]
[1 0 1]
[1 0 0]
[0 1 1]
[0 1 0]
[0 0 1]
[1 0 0 0 0 0]
[0 1 0 0 0 0]
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[0 0 0 0 1 0]
[0 0 0 0 0 1]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 1 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0]
{17, 11, 4, 23}

```

```

AY0 = Matrix(GF(2), \
    [[1,1,0],[1,0,1],[1,0,0],[0,1,1],[0,1,0],[0,0,1]])
AY1 = matrix.identity(6)
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1 0]

```

```

[1 0 1]
[1 0 0]
[0 1 1]
[0 1 0]
[0 0 1]
[1 0 0 0 0 0]
[0 1 0 0 0 0]
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[0 0 0 0 1 0]
[0 0 0 0 0 1]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 1 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0]
{17, 11, 4, 23}

```

```

AY0 = Matrix(GF(2), [[1,1,0],[1,0,1],[1,0,1],[0,1,1],[0,1,0]])
AY1 = matrix.identity(5)
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1 0]
[1 0 1]
[1 0 1]
[0 1 1]
[0 1 0]
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 0 1]

```

```

[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 0]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0]
{19, 11, 7}

```

```

AY0 = Matrix(GF(2), [[1,1,1],[1,0,1],[1,0,1],[0,1,1],[0,1,0]])
AY1 = matrix.identity(5)
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1 1]
[1 0 1]
[1 0 1]
[0 1 1]
[0 1 0]
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 0]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0]
{19, 11, 7}

```

```

AY0 = Matrix(GF(2), [[1,1,0],[1,0,1],[1,0,1],[0,1,1],[0,1,1]])
AY1 = matrix.identity(5)
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A

```

```
M.has_minor(L11)
```

```
# This tests for a (PG(3,2)_{-2})-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 13 and (M / S).rank() == 4:
        print S
        break
```

```
[1 1 0]
[1 0 1]
[1 0 1]
[0 1 1]
[0 1 1]
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0]
True
```

```
{8, 20, 4}
```

Lemma A.10. *If P_0 contains the columns*

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},$$

then either:

- (1) $Mw(\Phi) \not\subseteq EX(PG(3,2) \setminus e, L_{11})$ and $Mw(\Phi) \not\subseteq EX(PG(3,2)_{-2})$ or
- (2) P_0 is of the form

$$\left[\begin{array}{c|c|c|c} 1 \dots 1 & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 \\ 1 \dots 1 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 \\ \hline H_{1,1} & H_{1,0} & H_{0,1} & H_{0,0} \end{array} \right],$$

where each column of $H_{1,1}$ and $H_{0,0}$ has at most two nonzero entries, and where each column of $H_{0,1}$ and $H_{1,0}$ is a unit column or a zero column.

Proof. Suppose neither (1) nor (2) holds. Since (1) does not hold, Lemma A.8 implies that P_0 contains none of the following submatrices and also contains no column with five nonzero entries:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, since (2) does not hold, P_0 contains one of the following submatrices:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The computations below show that each of these submatrices results in a template to which virtually conforms a matroid which has $PG(3,2) \setminus e$ as a minor or which has both $PG(3,2)_{-2}$ and L_{11} as minors, contradicting the assumption that (1) does not hold. \square

```
AY0 = Matrix(GF(2), [[1,1,0],[1,1,0],[1,0,1],[1,0,1],[0,1,1]])
AY1 = matrix.identity(5)
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
```

```
M.has_minor(L11)
```

```
# This tests for a (PG(3,2)_{-2})-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 13 and (M / S).rank() == 4:
        print S
        break
```

```
[1 1 0]
[1 1 0]
[1 0 1]
[1 0 1]
[0 1 1]
```

```

[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0]
True

```

```
{8, 20, 4}
```

```

AY0 = Matrix(GF(2), [[1,1,1],[1,1,0],[1,0,1],[1,0,1],[0,1,1]])
AY1 = matrix.identity(5)
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1 1]
[1 1 0]
[1 0 1]
[1 0 1]
[0 1 1]
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0]
{10, 19, 14}

```

```

AY0 = Matrix(GF(2), [[1,1,1],[1,1,0],[1,0,1],[1,0,1],[0,1,0]])
AY1 = matrix.identity(5)
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

[1 1 1]
[1 1 0]
[1 0 1]
[1 0 1]
[0 1 0]
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 0]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0]
{10, 19, 14}

```

AY0 = Matrix(GF(2), [[1,1,1],[1,1,0],[1,0,1],[1,0,0],[0,1,1]])
AY1 = matrix.identity(5)
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

[1 1 1]

```

[1 1 0]
[1 0 1]
[1 0 0]
[0 1 1]
[1 0 0 0 0]
[0 1 0 0 0]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0]
{17, 18, 4}

```

```

AY0 = Matrix(GF(2), \
    [[1,1,1],[1,1,0],[1,0,1],[1,0,0],[0,1,0],[0,0,1]])
AY1 = matrix.identity(6)
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1 1]
[1 1 0]
[1 0 1]
[1 0 0]
[0 1 0]
[0 0 1]
[1 0 0 0 0 0]
[0 1 0 0 0 0]
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[0 0 0 0 1 0]
[0 0 0 0 0 1]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 0 1]

```

$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0]$
 $[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0]$
 $[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1]$
 $[1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0]$
 $[0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0]$
 $\{17, 4, 21, 20\}$

Lemma A.11. *If P_0 contains the columns*

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},$$

then either:

- (1) $Mw(\Phi) \notin EX(PG(3,2) \setminus e, L_{11})$ and $Mw(\Phi) \notin EX(PG(3,2)_{-2})$,
- (2) P_0 is of the form

$$\left[\begin{array}{c|c|c|c} 1 \cdots 1 & 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 \\ 1 \cdots 1 & 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \\ \hline H_{1,1} & H_{1,0} & H_{0,1} & H_{0,0} \end{array} \right],$$

where each column of $H_{1,1}$ and $H_{0,0}$ has at most two nonzero entries, and where each column of $H_{0,1}$ and $H_{1,0}$ is a unit column or a zero column, or

- (3) P_0 is of the following form, where each column of $[H_1|H_0]$ has at most two nonzero entries:

$$\left[\begin{array}{c|c} 1 \cdots 1 & 0 \cdots 0 \\ \hline H_1 & H_0 \end{array} \right].$$

Proof. Suppose neither (1), (2), nor (3) holds. First, suppose that P_0 contains any of the following submatrices:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

For each of these four submatrices, since (1) does not hold, Lemma A.8 implies that the column of P_0 containing the third column of the submatrix contains no nonzero entries except those in the submatrix. But then these submatrices are forbidden by Lemma A.9. Moreover, the following

submatrices are forbidden by Lemma A.8:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, since (2) does not hold, P_0 must contain the following submatrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since no column of P_0 can contain five nonzero entries, if this submatrix is contained in the following submatrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

then the column of P_0 containing the third column of this matrix contains no other nonzero entries. Therefore, this submatrix is forbidden by Lemma A.10. Thus, P_0 contains the following three columns:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, since (3) does not hold, P_0 contains one of the following submatrices, where x is either 0 or 1:

$$\begin{bmatrix} 1 & 1 & 1 & x \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & x \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & x \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & x \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The first of these four submatrices is forbidden by the computations below. If $x = 0$, the third and fourth of these submatrices is forbidden by Lemma A.8. The second submatrix is forbidden by

Lemma A.8 if the column of P_0 containing the fourth column contains an additional nonzero entry, but it is forbidden by Lemma A.9 otherwise. If $x = 1$, the second matrix is forbidden by Lemma A.10 and the fact that no column of P_0 can contain five nonzero entries. The third and fourth submatrices are forbidden by Lemma A.8. This completes the proof by contradiction. \square

```

AY0 = Matrix(GF(2), [[1,1,1,0],[1,1,0,1],[1,0,1,1],[0,1,1,1]])
AY1 = matrix.identity(4)
print AY0
print AY1
A=template_test(4, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break
[1 1 1 0]
[1 1 0 1]
[1 0 1 1]
[0 1 1 1]
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
[0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0]
[0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 1 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 1 1 1]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0]
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0]
{17, 12, 7}

```

```

AY0 = Matrix(GF(2), [[1,1,1,1],[1,1,0,1],[1,0,1,1],[0,1,1,1]])
AY1 = matrix.identity(4)
print AY0
print AY1
A=template_test(4, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

$[1 \ 1 \ 1 \ 1]$
 $[1 \ 1 \ 0 \ 1]$
 $[1 \ 0 \ 1 \ 1]$
 $[0 \ 1 \ 1 \ 1]$
 $[1 \ 0 \ 0 \ 0]$
 $[0 \ 1 \ 0 \ 0]$
 $[0 \ 0 \ 1 \ 0]$
 $[0 \ 0 \ 0 \ 1]$
 $[0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1]$
 $[0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1]$
 $[0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1]$
 $[0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1]$
 $[1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0]$
 $[0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0]$
 $[0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0]$
 $\{17, 12, 7\}$

Lemma A.12. *If P_0 contains the columns*

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},$$

then either:

- (1) $Mw(\Phi) \not\subseteq EX(PG(3, 2) \setminus e, L_{11})$ and $Mw(\Phi) \not\subseteq EX(PG(3, 2)_{-2})$
- (2) P_0 is of the form

$$\left[\begin{array}{c|c|c|c} 1 \cdots 1 & 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 \\ 1 \cdots 1 & 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \\ \hline H_{1,1} & H_{1,0} & H_{0,1} & H_{0,0} \end{array} \right],$$

where each column of $H_{1,1}$ and $H_{0,0}$ has at most two nonzero entries, and where each column of $H_{0,1}$ and $H_{1,0}$ is a unit column or a zero column, or

- (3) P_0 is of the following form, where each column of $[H_1|H_0]$ has at most two nonzero entries:

$$\left[\begin{array}{c|c} 1 \cdots 1 & 0 \cdots 0 \\ \hline H_1 & H_0 \end{array} \right].$$

Proof. Suppose neither (1), (2), nor (3) hold. Consider the following matrices:

$$\begin{aligned}
& \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \\
& \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

Since no column of P_0 can have five nonzero entries, the first two of these matrices are forbidden by Lemma A.10. The other six are forbidden by Lemma A.8. Now, for all but the first of these matrices, suppose it is not a submatrix of P_0 , but the matrix formed by removing the last row is a submatrix of P_0 . For the second matrix, this is impossible by Lemma A.11. For the third, fourth, fifth, and sixth matrices, this is impossible by Lemma A.9. For the seventh and eighth matrices, this is impossible by Lemma A.8. Thus, since (2) does not hold, P_0 must contain as a submatrix the matrix obtained by deleting either the third or fourth column from the matrix below.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

One of the computations used to prove Lemma A.11 shows that the above matrix itself is not a submatrix of P_0 .

Therefore, after swapping the second and third rows, we may assume without loss of generality that P_0 contains the following three columns:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}.$$

However, since (2) does not hold, P_0 must contain a fourth column resulting in P_0 containing one of the eight forbidden matrices above. This completes the proof by contradiction. \square

Lemma A.13. *If P_0 contains the submatrix*

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

then either:

- (1) $Mw(\Phi) \notin EX(PG(3,2) \setminus e, L_{11})$ and $Mw(\Phi) \notin EX(PG(3,2)_{-2})$ or
- (2) P_0 is of the form

$$\left[\begin{array}{c|c|c|c} 1 \dots 1 & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 \\ 1 \dots 1 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 \\ \hline H_{1,1} & H_{1,0} & H_{0,1} & H_{0,0} \end{array} \right],$$

where each column of $H_{1,1}$ and $H_{0,0}$ has at most two nonzero entries, and where each column of $H_{0,1}$ and $H_{1,0}$ is a unit column or a zero column.

Proof. Suppose neither (1) nor (2) holds. The following submatrices are forbidden from P_0 by Lemma A.8:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, since (2) does not hold and since no column of P_0 contains five nonzero entries, P_0 contains one of the following submatrices:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The computations below show that each of these submatrices results in a template to which virtually conforms a matroid which has $PG(3,2)\setminus e$ as a minor or which has both $PG(3,2)_{-2}$ and L_{11} as minors, contradicting the assumption that (1) does not hold. \square

```

AY0 = Matrix(GF(2), \
    [[1,1,0],[1,1,0],[1,0,1],[1,0,1],[0,1,1],[0,1,1]])
AY1 = matrix.identity(6)
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
M.has_minor(L11)
# This tests for a (PG(3,2)_{-2})-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 13 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1 0]
[1 1 0]
[1 0 1]
[1 0 1]
[0 1 1]
[0 1 1]
[1 0 0 0 0 0]
[0 1 0 0 0 0]
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[0 0 0 0 1 0]
[0 0 0 0 0 1]
[0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0]
True

```

{11, 23, 3, 7}

```

AY0 = Matrix(GF(2), \
    [[1,1,1],[1,1,0],[1,0,1],[1,0,1],[0,1,1],[0,1,0]])
AY1 = matrix.identity(6)
print AY0

```

```

print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1 1]
[1 1 0]
[1 0 1]
[1 0 1]
[0 1 1]
[0 1 0]
[1 0 0 0 0 0]
[0 1 0 0 0 0]
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[0 0 0 0 1 0]
[0 0 0 0 0 1]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 0]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0]
{10, 3, 22, 14}

```

```

AY0 = Matrix(GF(2), \
    [[1,1,1],[1,1,0],[1,0,1],[1,0,0],[0,1,1],[0,1,0]])
AY1 = matrix.identity(6)
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1 1]
[1 1 0]

```

$[1 \ 0 \ 1]$
 $[1 \ 0 \ 0]$
 $[0 \ 1 \ 1]$
 $[0 \ 1 \ 0]$
 $[1 \ 0 \ 0 \ 0 \ 0 \ 0]$
 $[0 \ 1 \ 0 \ 0 \ 0 \ 0]$
 $[0 \ 0 \ 1 \ 0 \ 0 \ 0]$
 $[0 \ 0 \ 0 \ 1 \ 0 \ 0]$
 $[0 \ 0 \ 0 \ 0 \ 1 \ 0]$
 $[0 \ 0 \ 0 \ 0 \ 0 \ 1]$
 $[0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1]$
 $[0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0]$
 $[0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1]$
 $[0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0]$
 $[0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1]$
 $[0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0]$
 $[1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0]$
 $[0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0]$
 $\{12, 11, 4, 22\}$

Lemma A.14. *If P_0 contains the submatrix*

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then either:

- (1) $Mw(\Phi) \not\subseteq EX(PG(3, 2) \setminus e, L_{11})$ and $Mw(\Phi) \not\subseteq EX(PG(3, 2)_{-2})$ or
- (2) P_0 is of the form

$$\left[\begin{array}{c|c|c|c} 1 \cdots 1 & 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 \\ 1 \cdots 1 & 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \\ \hline H_{1,1} & H_{1,0} & H_{0,1} & H_{0,0} \end{array} \right],$$

where each column of $H_{1,1}$ and $H_{0,0}$ has at most two nonzero entries, and where each column of $H_{0,1}$ and $H_{1,0}$ is a unit column or a zero column.

Proof. Suppose neither (1) nor (2) holds. The following submatrices are forbidden from P_0 by Lemma A.8:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The following submatrices are forbidden by Lemma A.13. Moreover, for each of these matrices, if the given matrix is not a submatrix of P_0 , then the submatrix obtained from this matrix by deleting the last row is forbidden from P_0 by Lemma A.10:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, since (2) does not hold, P_0 contains a submatrix obtained from the following matrix by deleting either the third or fourth column:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

This matrix itself is not a submatrix of P_0 by one of the computations used to prove Lemma A.11. Thus, after swapping the second and third rows, we may assume without loss of generality that P_0 contains the following submatrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since (2) does not hold and since we have already forbidden all other possibilities, P_0 must contain the following submatrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

but this contains a submatrix forbidden by a computation used to prove Lemma A.11. This completes the proof by contradiction. \square

Lemma A.15. *If $[P_1|P_0]$ contains any of the following submatrices, with the submatrix to the left of the vertical line contained in P_1 , with the column to the right of the vertical line contained in P_0 , and with either $x = 0$ or $x = 1$*

$$\begin{bmatrix} 1 & 1 & | & x \\ 1 & 0 & | & 1 \\ 0 & 1 & | & 1 \\ 0 & 0 & | & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & | & x \\ 1 & 0 & | & 0 \\ 0 & 1 & | & 1 \\ 0 & 0 & | & 1 \\ 0 & 0 & | & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & | & x \\ 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \\ 0 & 0 & | & 1 \\ 0 & 0 & | & 1 \end{bmatrix},$$

then $\mathcal{M}w(\Phi) \not\subseteq EX(PG(3,2)\backslash e)$.

Proof. See the computations below. \square

```

AY0 = Matrix(GF(2), [[0],[1],[1],[1]])
AY1 = Matrix(GF(2), \
    [[1,0,0,0,1,1],[0,1,0,0,1,0],[0,0,1,0,0,1],[0,0,0,1,0,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[0]
[1]
[1]
[1]
[1]
[1 0 0 0 1 1]
[0 1 0 0 1 0]
[0 0 1 0 0 1]
[0 0 0 1 0 0]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 1 1 0 0 0 1]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
{4, 21}

```

```

AY0 = Matrix(GF(2), [[1],[1],[1],[1]])
AY1 = Matrix(GF(2), \
    [[1,0,0,0,1,1],[0,1,0,0,1,0],[0,0,1,0,0,1],[0,0,0,1,0,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1]
[1]

```

```

[1]
[1]
[1 0 0 0 1 1]
[0 1 0 0 1 0]
[0 0 1 0 0 1]
[0 0 0 1 0 0]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1 1 0 0 0 1]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
{4, 21}

```

```

AY0 = Matrix(GF(2), [[0],[0],[1],[1],[1],])
AY1 = Matrix(GF(2), \
    [[1,0,0,0,0,1,1],[0,1,0,0,0,1,0],[0,0,1,0,0,0,1],[0,0,0,1,0,0,0],
    [0,0,0,0,1,0,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[0]
[0]
[1]
[1]
[1]
[1]
[1 0 0 0 0 1 1]
[0 1 0 0 0 1 0]
[0 0 1 0 0 0 1]
[0 0 0 1 0 0 0]
[0 0 0 0 1 0 0]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1]
{17, 13, 7}

```

```

AY0 = Matrix(GF(2), [[1],[0],[1],[1],[1]])
AY1 = Matrix(GF(2), \
    [[1,0,0,0,0,1,1],[0,1,0,0,0,1,0],[0,0,1,0,0,0,1],[0,0,0,1,0,0,0],
    [0,0,0,0,1,0,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1]
[0]
[1]
[1]
[1]
[1 0 0 0 0 1 1]
[0 1 0 0 0 1 0]
[0 0 1 0 0 0 1]
[0 0 0 1 0 0 0]
[0 0 0 0 1 0 0]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
{17, 13, 7}

```

```

AY0 = Matrix(GF(2), [[0],[0],[0],[1],[1],[1]])
AY1 = Matrix(GF(2), \
    [[1,0,0,0,0,0,1,1],[0,1,0,0,0,0,1,0],[0,0,1,0,0,0,0,1],
    [0,0,0,1,0,0,0,0],[0,0,0,0,1,0,0,0],[0,0,0,0,0,1,0,0]])
print AY0
print AY1
A=template_test(4, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):

```


$\{16, 10, 37, 38, 7\}$

in P_0 , and with either $x = 0$ or $x = 1$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right], \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right], \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array} \right]$$

then $Mw(\Phi) \not\subseteq EX(PG(3, 2) \setminus e)$.

Proof. See the computations below. \square

[1]

```

[0]
[1]
[1]
[1 0 0 0 1 1]
[0 1 0 0 1 1]
[0 0 1 0 1 0]
[0 0 0 1 0 1]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 1 1 1 1 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1 1 0 0 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
{11, 4}

```

```

AY0 = Matrix(GF(2), [[1],[0],[0],[1],[1]])
AY1 = Matrix(GF(2), \
    [[1,0,0,0,0,1,1],[0,1,0,0,0,1,1],[0,0,1,0,0,1,0],[0,0,0,1,0,0,1],
    [0,0,0,0,1,0,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1]
[0]
[0]
[1]
[1]
[1]
[1 0 0 0 0 1 1]
[0 1 0 0 0 1 1]
[0 0 1 0 0 1 0]
[0 0 0 1 0 0 1]
[0 0 0 0 1 0 0]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1 1 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 1 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1]
{9, 4, 17}

```

```

AY0 = Matrix(GF(2), [[1],[0],[0],[0],[1],[1]])
AY1 = Matrix(GF(2), \
    [[1,0,0,0,0,0,1,1],[0,1,0,0,0,0,1,1],[0,0,1,0,0,0,1,0],
     [0,0,0,1,0,0,0,1],[0,0,0,0,1,0,0,0],[0,0,0,0,0,1,0,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1]
[0]
[0]
[0]
[1]
[1]
[1 0 0 0 0 0 1 1]
[0 1 0 0 0 0 1 1]
[0 0 1 0 0 0 1 0]
[0 0 0 1 0 0 0 1]
[0 0 0 0 1 0 0 0]
[0 0 0 0 0 1 0 0]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 0]
[0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 1 1 1 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
{16, 9, 20, 13}

```

Lemma A.17. *If $[P_1|P_0]$ contains any of the following matrices as a submatrix, with the portion to the left of the vertical line contained in P_1 and the portion to the right contained in P_0 , then $\mathcal{M}_w(\Phi) \not\subseteq EX(PG(3,2)\backslash e)$:*

$$\left[\begin{array}{c|cc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right], \left[\begin{array}{c|cc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right], \left[\begin{array}{c|cc} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right], \left[\begin{array}{c|cc} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right],$$

Proof. See the computations below. \square

```
AY0 = Matrix(GF(2), [[1,0],[0,1],[1,1],[1,1]])
AY1 = Matrix(GF(2), [[1,0,0,0,1],[0,1,0,0,1],[0,0,1,0,0], \
    [0,0,0,1,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break
```

```
[1 0]
[0 1]
[1 1]
[1 1]
[1 0 0 0 1]
[0 1 0 0 1]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 1 1 1 0]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 1 1 0 1]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
{10, 14}
```

```
AY0 = Matrix(GF(2), [[1,0],[0,1],[1,1],[1,0],[0,1]])
AY1 = Matrix(GF(2), [[1,0,0,0,0,1],[0,1,0,0,0,1],[0,0,1,0,0,0], \
    [0,0,0,1,0,0],[0,0,0,0,1,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break
```

```
[1 0]
[0 1]
```

```

[1 1]
[1 0]
[0 1]
[1 0 0 0 1]
[0 1 0 0 1]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 0 1]
[0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0]
[0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 1 1 1 0 1]
[0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 1 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
{4, 22, 14}

```

```

AY0 = Matrix(GF(2), [[1,1],[1,0],[1,1],[1,1]])
AY1 = Matrix(GF(2), [[1,0,0,0,1],[0,1,0,0,1],[0,0,1,0,0], \
    [0,0,0,1,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1]
[1 0]
[1 1]
[1 1]
[1 0 0 1]
[0 1 0 1]
[0 0 1 0]
[0 0 0 1]
[0 0 0 1 1 0 0 0 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 1 1 1 0 0 0 0 0 1 1 1 0]
[0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
{10, 14}

```

```

AY0 = Matrix(GF(2), [[1,1],[1,0],[1,0],[1,1],[0,1]])
AY1 = Matrix(GF(2), [[1,0,0,0,0,1],[0,1,0,0,0,1],[0,0,1,0,0,0], \
    [0,0,0,1,0,0],[0,0,0,0,1,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1]
[1 0]
[1 0]
[1 1]
[0 1]
[1 0 0 0 0 1]
[0 1 0 0 0 1]
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[0 0 0 0 1 0]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0]
[0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
{17, 4, 21}

```

Lemma A.18. *If $[P_1|P_0]$ contains either of the matrices below as a submatrix, with the portion to the left of the vertical line contained in P_1 and the portion to the right contained in P_0 , then $\mathcal{M}_w(\Phi) \not\subseteq EX(PG(3,2)\backslash e)$:*

$$\left[\begin{array}{c|ccc} 1 & 1 & 1 & \\ 1 & 1 & 0 & \\ 1 & 0 & 1 & \\ 0 & 1 & 1 & \end{array} \right], \left[\begin{array}{c|ccc} 1 & 1 & 1 & \\ 1 & 1 & 0 & \\ 1 & 0 & 1 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array} \right].$$

Proof. See the computations below. \square

```

AY0 = Matrix(GF(2), [[1,1],[1,0],[0,1],[1,1]])
AY1 = Matrix(GF(2), [[1,0,0,0,1],[0,1,0,0,1],[0,0,1,0,1], \
    [0,0,0,1,0]])

```

```

print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1]
[1 0]
[0 1]
[1 1]
[1 0 0 0 1]
[0 1 0 0 1]
[0 0 1 0 1]
[0 0 0 1 0]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 1 1 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1 1 1 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
{4, 14}

```

```

AY0 = Matrix(GF(2), [[1,1],[1,0],[0,1],[1,0],[0,1]])
AY1 = Matrix(GF(2), [[1,0,0,0,0,1],[0,1,0,0,0,1],[0,0,1,0,0,1], \
    [0,0,0,1,0,0],[0,0,0,0,1,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)\e)-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 14 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1]
[1 0]
[0 1]
[1 0]
[0 1]
[1 0 0 0 0 1]
[0 1 0 0 0 1]
[0 0 1 0 0 1]

```

```

[0 0 0 1 0 0]
[0 0 0 0 1 0]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 1 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
{17, 4, 14}

```

Lemma A.19. *If P_1 contains either of the following submatrices, then $Mw(\Phi) \notin EX(PG(3,2)_{-2})$.*

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Proof. See the computations below. \square

```

AY0 = Matrix(GF(2),3,0)
AY1 = Matrix(GF(2), [[1,0,0,1,1],[0,1,0,1,0],[0,0,1,0,1]])
print AY0
print AY1
A=template_test(4, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)_{-2})-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 13 and (M / S).rank() == 4:
        print S
        break

[]
[1 0 0 1 1]
[0 1 0 1 0]
[0 0 1 0 1]
[0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1]
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0]
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1]
{0, 12}

```

```

AY0 = Matrix(GF(2),4,0)

```

```

AY1 = Matrix(GF(2), \
    [[1,0,0,0,1,1],[0,1,0,0,1,1],[0,0,1,0,1,0],[0,0,0,1,0,1]])
print AY0
print AY1
A=template_test(4, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)_{-2})-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 13 and (M / S).rank() == 4:
        print S
        break

[]
[1 0 0 0 1 1]
[0 1 0 0 1 1]
[0 0 1 0 1 0]
[0 0 0 1 0 1]
[0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1]
[0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0]
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1]
{0, 16, 6}

```

Lemma A.20. *If $[P_1|P_0]$ contains either of the following submatrices, with the portion to the left of the vertical line contained in P_1 and the portion to the right contained in P_0 , then $\mathcal{M}_w(\Phi) \not\subseteq EX(PG(3,2)_{-2})$:*

$$\left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{array} \right], \left[\begin{array}{c|c} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{array} \right].$$

Proof. See the computations below. \square

```

AY0 = Matrix(GF(2), [[1],[1],[1],[1]])
AY1 = Matrix(GF(2), \
    [[1,0,0,0,1],[0,1,0,0,1],[0,0,1,0,0],[0,0,0,1,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)_{-2})-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 13 and (M / S).rank() == 4:

```

```

        print S
        break

[1]
[1]
[1]
[1]
[1 0 0 0 1]
[0 1 0 0 1]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
{18, 4}

AY0 = Matrix(GF(2), [[1],[0],[1],[1]])
AY1 = Matrix(GF(2), \
    [[1,0,0,0,1],[0,1,0,0,1],[0,0,1,0,0],[0,0,0,1,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)-{-2})-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 13 and (M / S).rank() == 4:
        print S
        break

[1]
[0]
[1]
[1]
[1 0 0 0 1]
[0 1 0 0 1]
[0 0 1 0 0]
[0 0 0 1 0]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1 1 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
{18, 7}

```

Lemma A.21. If $[P_1|P_0]$ contains the submatrix below, with the portion to the left of the vertical line contained in P_1 and the portion to the right contained in P_0 , then $\mathcal{M}_w(\Phi) \notin EX(PG(3,2)_{-2})$:

$$\left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{array} \right]$$

Proof. See the computations below. \square

```

AY0 = Matrix(GF(2), [[1],[1],[0],[1]])
AY1 = Matrix(GF(2), \
    [[1,0,0,0,1],[0,1,0,0,1],[0,0,1,0,1],[0,0,0,1,0]])
print AY0
print AY1
A=template_test(3, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)_{-2})-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 13 and (M / S).rank() == 4:
        print S
        break

```

```

[1]
[1]
[0]
[1]
[1 0 0 0 1]
[0 1 0 0 1]
[0 0 1 0 1]
[0 0 0 1 0]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1 1 0]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
{4, 14}

```

Lemma A.22. If P_0 contains the submatrix below, then $\mathcal{M}_w(\Phi) \notin EX(PG(3,2)_{-2})$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Proof. See the computation below. \square

```

AY0 = Matrix(GF(2), [[1,1,1],[1,0,1],[1,1,0],[0,1,1]])
AY1 = matrix.identity(4)
print AY0
print AY1
A=template_test(4, AY1, AY0)
M=Matroid(field=GF(2), matrix=A)
print A
# This tests for a (PG(3,2)_{-2})-minor.
for S in Subsets(M.groundset(), M.rank() - 4):
    if len((M / S).simplify()) >= 13 and (M / S).rank() == 4:
        print S
        break

```

```

[1 1 1]
[1 0 1]
[1 1 0]
[0 1 1]
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
[0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1]
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 1 1]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0]
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0]
{17, 12, 7}

```

```

version()
'SageMath version 7.3, Release Date: 2016-08-04'

```

Appendix B. Even-cut computations

10/1/2016

The following functions allow us to test if a given matroid M is even-cut. If not, it allows us to find a minor of M that is an excluded minor for the class of even-cut matroids.

```
from sage.matroids.advanced import *
def is_even_cut(M, certificate=False):
    e = newlabel(M.groundset())
    for N in M.linear_extensions(e):
        if (N / e).dual().is_graphic():
            if certificate:
                return (True, N, e)
            else:
                return True
    return False

def is_even_cut_excluded_minor(M):
    if is_even_cut(M):
        return False
    for e in M.groundset():
        if not is_even_cut(M \ e) or not is_even_cut(M / e):
            # not minimal
            return False
    return True

def make_even_cut_excluded_minor(M):
    if is_even_cut(M):
        raise ValueError("Need non-even-cut input")
    else:
        return make_even_cut_excluded_minor_rec(M, [], [])

def make_even_cut_excluded_minor_rec(M, delset, conset):
    # get an element of the ground set:
    for e in M.groundset():
        if not is_even_cut(M / e):
            return make_even_cut_excluded_minor_rec(M / e, delset, \
conset + [e])
```

```

        if not is_even_cut(M \ e):
            return make_even_cut_excluded_minor_rec(M \ e, delset + \
[e], conset)
        # If we reach this point, M must be an excluded minor itself!
        return (M, delset, conset)

```

Lemma B.1. *The matroid $M(K_6)$ is an excluded minor for the class of even-cut matroids.*
Proof. See the computations below. \square

```

is_even_cut(matroids.CompleteGraphic(6))
False

```

```

is_even_cut(matroids.CompleteGraphic(6) \ 1) and is_even_cut(\
matroids.CompleteGraphic(6) / 1)
True

```

Lemma B.2. *The matroid H_{12}^* is an excluded minor for the class of even-cut matroids.*

Proof. See the computations below. The empty brackets in the result indicate that to obtain an excluded minor from H_{12}^* , there are no edges that must be contracted or deleted. Therefore, H_{12}^* is itself an excluded minor. \square

```

A = Matrix(GF(2), \
    [[1,0,1,0,1,0,1,0,1,0,1,0],[0,0,0,0,1,1,1,1,0,0,0,0],
    \
    [0,0,0,0,1,1,0,0,1,1,0,0],[1,1,0,0,0,0,0,0,0,0,1,1],
    [0,0,1,1,0,0,1,1,0,0,1,1]])
print A

H12=Matroid(field=GF(2), matrix=A)
make_even_cut_excluded_minor(H12.dual())
[1 0 1 0 1 0 1 0 1 0 1 0]
[0 0 0 0 1 1 1 1 0 0 0 0]
[0 0 0 0 1 1 0 0 1 1 0 0]
[1 1 0 0 0 0 0 0 0 0 1 1]
[0 0 1 1 0 0 1 1 0 0 1 1]
(Binary matroid of rank 7 on 12 elements, type (3, 0), [], [])

```

The following function returns a matrix of maximum size virtually conforming to the template $\Psi = (\{1\}, \emptyset, D, \emptyset, Y_0, Y_1, A_1, \{0, \bar{x}\}, \emptyset)$, where A_1 is of the form $[I|A_{Y_1}|B_{Y_1}|A_{Y_0}|B_{Y_0}]$, as given in the main paper in the proof of Theorem 4.6.

```

def matrix_from_template(r, Y1I, Y1A, Y1B, Y0A, Y0B, B_Jrows):
    print "Template with C empty, Y1 consisting of matrices I, A, B,\
with " , B_Jrows, " all-one rows below B and then zeroes below B\
"
    print "Y0 consisting of matrices A, B with " , B_Jrows, " all-one\

```

```

rows below B and then zeroes"
print "I = "
print Y1I
print "A_Y1 = "
print Y1A
print "B_Y1 = "
print Y1B
print "A_Y0 = "
print Y0A
print "B_Y0 = "
print Y0B
c = Y1I.nrows()
A = Matrix(GF(2), r + c, binomial(r+1,2) + (r+1) * Y1I.ncols() + \
(r+1) * Y1A.ncols() + (r+1) * Y1B.ncols() + Y0A.ncols() + Y0B.\
ncols())
i = 0
for j in range(c,r+c):
    A[j,i] = 1
    i = i + 1
for j in range(c, r-1 + c):
    for k in range(j+1,r+c):
        A[j,i] = 1
        A[k,i] = 1
        i = i + 1
for j in range(Y1I.ncols()):
    for k in range(c):
        A[k,i] = Y1I[k,j]
    i = i + 1
    for l in range(r):
        for k in range(c):
            A[k,i] = Y1I[k,j]
        A[c+l, i] = 1
        i = i + 1
for j in range(Y1A.ncols()):
    for k in range(c):
        A[k,i] = Y1A[k,j]
    i = i + 1
    for l in range(r):
        for k in range(c):
            A[k,i] = Y1A[k,j]
        A[c+l, i] = 1
        i = i + 1
for j in range(Y1B.ncols()):
    for k in range(c):
        A[k,i] = Y1B[k,j]
    for k in range(B_Jrows):
        A[c+k,i] = A[c+k,i] + 1

```

```

        i = i + 1
        for l in range(r):
            for k in range(c):
                A[k,i] = Y1B[k,j]
                A[c+l, i] = 1
            for k in range(B_Jrows):
                A[c+k,i] = A[c+k,i] + 1
            i = i + 1
    for j in range(Y0A.ncols()):
        for k in range(c):
            A[k,i] = Y0A[k,j]
        i = i + 1
    for j in range(Y0B.ncols()):
        for k in range(c):
            A[k,i] = Y0B[k,j]
        for k in range(c, c+B_Jrows):
            A[k,i] = 1
        i = i + 1
    return A

```

Lemma B.3. *If B_{Y_1} contains the submatrix $[1, 0]$, then $\mathcal{M}_w(\Psi) \not\subseteq EX(H_{12})$.*

Proof. See the computation below. \square

```

A = matrix_from_template(4, identity_matrix(GF(2), 1), Matrix(GF(2), \
    []), Matrix(GF(2), [[1, 0]]), Matrix(GF(2), []), Matrix(GF(2), []), \
    3)
print A
M = Matroid(A)
M.has_minor(H12)

```

Template with C empty, Y1 consisting of matrices I, A, B, with 3 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 3 all-one rows below B and then zeroes

```

I =
[1]
A_Y1 =
[]
B_Y1 =
[1 0]
A_Y0 =
[]
B_Y0 =
[]
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 0 0 0 0 0]
[1 0 0 0 1 1 1 0 0 0 0 1 0 0 0 1 0 1 1 1 0 1 1 1]
[0 1 0 0 1 0 0 1 1 0 0 0 1 0 0 1 1 0 1 1 1 1 0 1 1]
[0 0 1 0 0 1 0 1 0 1 0 0 0 1 0 1 1 1 0 1 1 1 1 0 1]
[0 0 0 1 0 0 1 0 1 1 0 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1]

```

True

Lemma B.4. *If $[A_{Y_1}|B_{Y_1}]$ contains any of the following submatrices, with the column to the left of the vertical line contained in A_{Y_1} , and the column to the right of the vertical line contained in B_{Y_1} , then $Mw(\Psi) \notin EX(H_{12})$.*

$$\left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 1 \end{array} \right], \left[\begin{array}{c|c} 1 & 0 \\ \hline 1 & 0 \end{array} \right], \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 0 \end{array} \right]$$

Proof. See the computations below. \square

```
A = matrix_from_template(3,identity_matrix(GF(2),2), Matrix(GF(2), \
    [[1],[1]]), Matrix(GF(2), [[1],[1]]), Matrix(GF(2), []), Matrix(\
    GF(2),[]), 3)
```

```
print A
```

```
M = Matroid(A)
```

```
M.has_minor(H12)
```

Template with C empty, Y1 consisting of matrices I, A, B, with 3 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 3 all-one rows below B and then zeroes

```
I =
```

```
[1 0]
```

```
[0 1]
```

```
A_Y1 =
```

```
[1]
```

```
[1]
```

```
B_Y1 =
```

```
[1]
```

```
[1]
```

```
A_Y0 =
```

```
[]
```

```
B_Y0 =
```

```
[]
```

```
[0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 1 1 1]
```

```
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1]
```

```
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 1 0 1]
```

```
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1 1 0]
```

```
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 1 1 0]
```

```
True
```

```
A = matrix_from_template(3,identity_matrix(GF(2),2), Matrix(GF(2), \
    [[1],[1]]), Matrix(GF(2), [[0],[0]]), Matrix(GF(2), []), Matrix(\
    GF(2),[]), 3)
```

```
print A
```

```
M = Matroid(A)
```

```
M.has_minor(H12)
```

```
Template with C empty, Y1 consisting of matrices I, A, B, with 3 all-one rows below B  
and then zeroes below B
```

```
Y0 consisting of matrices A, B with 3 all-one rows below B and then zeroes
```

```
I =
```

```
[1 0]
```

```
[0 1]
```

```
A_Y1 =
```

```
[1]
```

```
[1]
```

```
B_Y1 =
```

```
[0]
```

```
[0]
```

```
A_Y0 =
```

```
[]
```

```
B_Y0 =
```

```
[]
```

```
[0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0]
```

```
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 0 0 0 0]
```

```
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 1 0 1 1]
```

```
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1 1 0 1]
```

```
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 1 1 1 0]
```

```
True
```

```
A = matrix_from_template(3,identity_matrix(GF(2),2), Matrix(GF(2), \  
    [[1],[1]]), Matrix(GF(2), [[1],[0]]), Matrix(GF(2), []), Matrix(\  
    GF(2),[]), 3)
```

```
print A
```

```
M = Matroid(A)
```

```
M.has_minor(H12)
```

```
Template with C empty, Y1 consisting of matrices I, A, B, with 3 all-one rows below B  
and then zeroes below B
```

```
Y0 consisting of matrices A, B with 3 all-one rows below B and then zeroes
```

```
I =
```

```
[1 0]
```

```
[0 1]
```

```
A_Y1 =
```

```
[1]
```

```
[1]
```

```
B_Y1 =
```

```
[1]
```

```
[0]
```

```
A_Y0 =
```

```
[]
```

```
B_Y0 =
```

```
[]
```

```

[0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 1 1 1]
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 0 0 0]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 1 0 1]
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1 1 0]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 1 1 0]
True

```

Lemma B.5. *If A_{Y_1} contains any of the following submatrices, then $\mathcal{Mw}(\Psi) \notin EX(H_{12})$.*

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Proof. See the computations below. \square

```

A = matrix_from_template(2, identity_matrix(GF(2), 3), Matrix(GF(2), \
    [[1,1],[1,1],[0,1]]), Matrix(GF(2), []), Matrix(GF(2), []), \
    Matrix(GF(2), []), 0)

```

```
print A
```

```
M = Matroid(A)
```

```
M.has_minor(H12)
```

Template with C empty, Y1 consisting of matrices I, A, B, with 0 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 0 all-one rows below B and then zeroes

```
I =
```

```
[1 0 0]
```

```
[0 1 0]
```

```
[0 0 1]
```

```
A_Y1 =
```

```
[1 1]
```

```
[1 1]
```

```
[0 1]
```

```
B_Y1 =
```

```
[]
```

```
A_Y0 =
```

```
[]
```

```
B_Y0 =
```

```
[]
```

```
[0 0 0 1 1 1 0 0 0 0 0 0 1 1 1 1 1 1]
```

```
[0 0 0 0 0 0 1 1 1 0 0 0 1 1 1 1 1]
```

```
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1]
```

```
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
```

```
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1]
```

```
True
```

```
A = matrix_from_template(2,identity_matrix(GF(2),3), Matrix(GF(2), \
    [[1,0],[1,1],[0,1]]), Matrix(GF(2), []), Matrix(GF(2), []), \
    Matrix(GF(2),[]), 0)
```

```
print A
```

```
M = Matroid(A)
```

```
M.has_minor(H12)
```

Template with C empty, Y1 consisting of matrices I, A, B, with 0 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 0 all-one rows below B and then zeroes

```
I =
```

```
[1 0 0]
```

```
[0 1 0]
```

```
[0 0 1]
```

```
A_Y1 =
```

```
[1 0]
```

```
[1 1]
```

```
[0 1]
```

```
B_Y1 =
```

```
[]
```

```
A_Y0 =
```

```
[]
```

```
B_Y0 =
```

```
[]
```

```
[0 0 0 1 1 1 0 0 0 0 0 0 1 1 1 0 0 0]
```

```
[0 0 0 0 0 0 1 1 1 0 0 0 1 1 1 1 1]
```

```
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1]
```

```
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
```

```
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1]
```

```
True
```

```
A = matrix_from_template(2,identity_matrix(GF(2),4), Matrix(GF(2), \
    [[1,0],[1,0],[0,1],[0,1]]), Matrix(GF(2), []), Matrix(GF(2), []), \
    Matrix(GF(2),[]), 0)
```

```
print A
```

```
M = Matroid(A)
```

```
M.has_minor(H12)
```

Template with C empty, Y1 consisting of matrices I, A, B, with 0 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 0 all-one rows below B and then zeroes

```
I =
```

```
[1 0 0 0]
```

```
[0 1 0 0]
```

```
[0 0 1 0]
```

```
[0 0 0 1]
```

```
A_Y1 =
```

```
[1 0]
```

```

[1 0]
[0 1]
[0 1]
B_Y1 =
[]
A_Y0 =
[]
B_Y0 =
[]
[0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0]
[0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1 1 0 0 0]
[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1 1]
[1 0 1 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0]
[0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 0 0 1]
True

```

Lemma B.6. *If $[A_{Y_1}|B_{Y_0}]$ contains any of the following submatrices, with the column to the left of the vertical line contained in A_{Y_1} , and the column to the right of the vertical line contained in B_{Y_0} , then $Mw(\Psi) \not\subseteq EX(H_{12})$.*

$$\left[\begin{array}{c|c} 1 & 0 \\ 1 & 0 \end{array} \right], \left[\begin{array}{c|c} 1 & 1 \\ 1 & 0 \end{array} \right], \left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \end{array} \right]$$

Proof. See the computations below. \square

```

A = matrix_from_template(3,identity_matrix(GF(2),2), Matrix(GF(2), \
    [[1],[1]]), Matrix(GF(2), []), Matrix(GF(2), []), Matrix(GF(2)\
    ,[[0],[0]]), 3)
print A
M = Matroid(A)
M.has_minor(H12)

```

Template with C empty, Y1 consisting of matrices I, A, B, with 3 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 3 all-one rows below B and then zeroes

```

I =
[1 0]
[0 1]
A_Y1 =
[1]
[1]
B_Y1 =
[]
A_Y0 =
[]
B_Y0 =

```

```

[0]
[0]
[0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 0]
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 0]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 1]
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 1]
True

```

```

A = matrix_from_template(3,identity_matrix(GF(2),2), Matrix(GF(2), \
    [[1],[1]]), Matrix(GF(2), []), Matrix(GF(2), []), Matrix(GF(2)\
    ,[[1],[0]]), 3)
print A
M = Matroid(A)
M.has_minor(H12)
Template with C empty, Y1 consisting of matrices I, A, B, with 3 all-one rows below B
and then zeroes below B
Y0 consisting of matrices A, B with 3 all-one rows below B and then zeroes
I =
[1 0]
[0 1]
A_Y1 =
[1]
[1]
B_Y1 =
[]
A_Y0 =
[]
B_Y0 =
[1]
[0]
[0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 1]
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 0]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 1]
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 1]
True

```

```

A = matrix_from_template(3,identity_matrix(GF(2),2), Matrix(GF(2), \
    [[1],[1]]), Matrix(GF(2), []), Matrix(GF(2), []), Matrix(GF(2)\
    ,[[1],[1]]), 3)
print A
M = Matroid(A)
M.has_minor(H12)
Template with C empty, Y1 consisting of matrices I, A, B, with 3 all-one rows below B

```

```

and then zeroes below B
Y0 consisting of matrices A, B with 3 all-one rows below B and then zeroes
I =
[1 0]
[0 1]
A_Y1 =
[1]
[1]
B_Y1 =
[]
A_Y0 =
[]
B_Y0 =
[1]
[1]
[0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 1]
[0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 1]
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 1]
True

```

Lemma B.7. *If $[B_{Y_1}|A_{Y_0}]$ contains any of the following submatrices, with the column to the left of the vertical line contained in B_{Y_1} , and the column to the right of the vertical line contained in A_{Y_0} , then $Mw(\Psi) \not\subseteq EX(H_{12})$.*

$$\left[\begin{array}{c|c} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{array} \right], \left[\begin{array}{c|c} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{array} \right], \left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{array} \right], \left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array} \right]$$

Proof. See the computations below. \square

```

A = matrix_from_template(3, identity_matrix(GF(2), 3), Matrix(GF(2), \
    []), Matrix(GF(2), [[0],[0],[0]]), Matrix(GF(2), [[1],[1],[1]]), \
    Matrix(GF(2), []), 3)

```

```
print A
```

```
M = Matroid(A)
```

```
M.has_minor(H12)
```

Template with C empty, Y1 consisting of matrices I, A, B, with 3 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 3 all-one rows below B and then zeroes

```

I =
[1 0 0]
[0 1 0]
[0 0 1]
A_Y1 =
[]

```

```

B_Y1 =
[0]
[0]
[0]
A_Y0 =
[1]
[1]
[1]
B_Y0 =
[]
[0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1]
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 1 0 1 1 0]
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1 1 0 1 0]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 1 1 1 0 0]
True

```

```

A = matrix_from_template(3,identity_matrix(GF(2),3), Matrix(GF(2), \
    []), Matrix(GF(2), [[1],[0],[0]]), Matrix(GF(2), [[1],[1],[1]]), \
    Matrix(GF(2),[]), 3)

```

```
print A
```

```
M = Matroid(A)
```

```
M.has_minor(H12)
```

Template with C empty, Y1 consisting of matrices I, A, B, with 3 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 3 all-one rows below B and then zeroes

```

I =
[1 0 0]
[0 1 0]
[0 0 1]
A_Y1 =
[]
B_Y1 =
[1]
[0]
[0]
A_Y0 =
[1]
[1]
[1]
B_Y0 =
[]
[0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 1 1 1 1 1]
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 1 0 1 1 0]

```

```
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1 1 0 1 0]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 1 1 1 0 0]
```

```
True
```

```
A = matrix_from_template(3,identity_matrix(GF(2),3), Matrix(GF(2), \
    []), Matrix(GF(2), [[1],[1],[0]]), Matrix(GF(2), [[1],[1],[1]]), \
    Matrix(GF(2),[]), 3)
```

```
print A
```

```
M = Matroid(A)
```

```
M.has_minor(H12)
```

Template with C empty, Y1 consisting of matrices I, A, B, with 3 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 3 all-one rows below B and then zeroes

```
I =
```

```
[1 0 0]
```

```
[0 1 0]
```

```
[0 0 1]
```

```
A_Y1 =
```

```
[]
```

```
B_Y1 =
```

```
[1]
```

```
[1]
```

```
[0]
```

```
A_Y0 =
```

```
[1]
```

```
[1]
```

```
[1]
```

```
B_Y0 =
```

```
[]
```

```
[0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 1 1 1 1 1]
```

```
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 1]
```

```
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1]
```

```
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 1 0 1 1 0]
```

```
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1 1 0 1 0]
```

```
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 1 1 1 0 0]
```

```
True
```

```
A = matrix_from_template(2,identity_matrix(GF(2),3), Matrix(GF(2), \
    []), Matrix(GF(2), [[1],[1],[1]]), Matrix(GF(2), [[1],[1],[1]]), \
    Matrix(GF(2),[]), 2)
```

```
print A
```

```
M = Matroid(A)
```

```
M.has_minor(H12)
```

Template with C empty, Y1 consisting of matrices I, A, B, with 2 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 2 all-one rows below B and then zeroes

```
I =
```

```

[1 0 0]
[0 1 0]
[0 0 1]
A_Y1 =
[]
B_Y1 =
[1]
[1]
[1]
A_Y0 =
[1]
[1]
[1]
B_Y0 =
[]
[0 0 0 1 1 1 0 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 0 1 1 1 0 0 0 1 1 1 1]
[0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1]
[1 0 1 0 1 0 0 1 0 0 1 0 1 0 1 0]
[0 1 1 0 0 1 0 0 1 0 0 1 1 1 0 0]
True

```

Lemma B.8. *If $[B_{Y_1}|B_{Y_0}]$ contains any of the following submatrices, with the column to the left of the vertical line contained in B_{Y_1} , and the column to the right of the vertical line contained in B_{Y_0} , then $Mw(\Psi) \not\subseteq EX(H_{12})$.*

$$\left[\begin{array}{c|c} 0 & 1 \\ 0 & 1 \end{array} \right], \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{c|c} 1 & 0 \\ 1 & 0 \end{array} \right]$$

Proof. See the computations below. \square

```

A = matrix_from_template(4,identity_matrix(GF(2),2), Matrix(GF(2), \
    []), Matrix(GF(2), [[0],[0]]), Matrix(GF(2), []), Matrix(GF(2)\
    ,[[1],[1]]), 4)

```

```

print A

```

```

M = Matroid(A)

```

```

N=(M/ 11).simplify()

```

```

N.has_minor(H12)

```

Template with C empty, Y1 consisting of matrices I, A, B, with 4 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 4 all-one rows below B and then zeroes

```

I =

```

```

[1 0]

```

```

[0 1]

```

```

A_Y1 =

```

```

[]

```

```

B_Y1 =

```

```

[0]
[0]
A_Y0 =
[]
B_Y0 =
[1]
[1]
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0 0 0 0 0 0 0 0 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0 0 0 0 0 1]
[1 0 0 0 1 1 1 0 0 0 0 1 0 0 0 0 1 0 0 0 1 0 1 1 1]
[0 1 0 0 1 0 0 1 1 0 0 0 1 0 0 0 0 1 0 0 1 1 0 1 1]
[0 0 1 0 0 1 0 1 0 1 0 0 0 1 0 0 0 0 1 0 1 1 1 0 1]
[0 0 0 1 0 0 1 0 1 1 0 0 0 0 1 0 0 0 0 1 1 1 1 0 1]
True

```

```

A = matrix_from_template(3,identity_matrix(GF(2),2), Matrix(GF(2), \
    []), Matrix(GF(2), [[1],[0]]), Matrix(GF(2), []), Matrix(GF(2)\
    ,[[0],[1]]), 2)
print A
M = Matroid(A)
M.has_minor(H12)

```

Template with C empty, Y1 consisting of matrices I, A, B, with 2 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 2 all-one rows below B and then zeroes

```

I =
[1 0]
[0 1]
A_Y1 =
[]
B_Y1 =
[1]
[0]
A_Y0 =
[]
B_Y0 =
[0]
[1]
[0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 0]
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 1 0 1 1 1]
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 1 1 0 1 1]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 0]
True

```

```

A = matrix_from_template(3,identity_matrix(GF(2),2), Matrix(GF(2), \

```

```

    []), Matrix(GF(2), [[1],[1]]), Matrix(GF(2), []), Matrix(GF(2)\
    ,[[0],[0]]), 3)

```

```

print A
M = Matroid(A)
M.has_minor(H12)

```

Template with C empty, Y1 consisting of matrices I, A, B, with 3 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 3 all-one rows below B and then zeroes

```

I =
[1 0]
[0 1]
A_Y1 =
[]
B_Y1 =
[1]
[1]
[1]
A_Y0 =
[]
B_Y0 =
[0]
[0]
[0]
[0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 0]
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 0]
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 1 0 1 1 1]
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 1 1 0 1 1]
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 1 1 1 0 1]
True

```

Lemma B.9. *If A_{Y_0} contains the following submatrix, then $\mathcal{M}w(\Psi) \notin EX(H_{12})$.*

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Proof. See the computation below. \square

```

A = matrix_from_template(1, identity_matrix(GF(2), 6), Matrix(GF(2), \
    []), Matrix(GF(2), []), Matrix(GF(2), \
    [[1,0,1,0],[1,1,0,0],[1,1,1,1],[1,1,1,1],[1,1,1,1],[0,1,1,0]]), \
    Matrix(GF(2), []), 0)
print A
M = Matroid(A)
M.has_minor(H12)

```

Template with C empty, Y1 consisting of matrices I, A, B, with 0 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 0 all-one rows below B and then zeroes

```
I =
[1 0 0 0 0 0]
[0 1 0 0 0 0]
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[0 0 0 0 1 0]
[0 0 0 0 0 1]
A_Y1 =
[]
B_Y1 =
[]
A_Y0 =
[1 0 1 0]
[1 1 0 0]
[1 1 1 1]
[1 1 1 1]
[1 1 1 1]
[0 1 1 0]
B_Y0 =
[]
[0 1 1 0 0 0 0 0 0 0 0 0 1 0 1 0]
[0 0 0 1 1 0 0 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 1 1 0 0 0 0 0 0 1 1 1]
[0 0 0 0 0 0 0 1 1 0 0 0 0 1 1 1]
[0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 1]
[0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 1]
[1 0 1 0 1 0 1 0 1 0 1 0 1 0 0 0]
True
```

Lemma B.10. *If B_{Y_0} contains the following submatrix, then $\mathcal{M}w(\Psi) \notin EX(H_{12})$.*

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Proof. See the computation below. \square

```
A = matrix_from_template(3, identity_matrix(GF(2), 2), Matrix(GF(2), \
    []), Matrix(GF(2), []), Matrix(GF(2), []), Matrix(GF(2)\
    , [[1, 1, 0, 0], [1, 0, 1, 0]]), 3)
print A
M = Matroid(A)
M.has_minor(H12)
```

Template with C empty, Y1 consisting of matrices I, A, B, with 3 all-one rows below B and then zeroes below B

Y0 consisting of matrices A, B with 3 all-one rows below B and then zeroes

```
I =  
[1 0]  
[0 1]  
A_Y1 =  
[]  
B_Y1 =  
[]  
A_Y0 =  
[]  
B_Y0 =  
[1 1 0 0]  
[1 0 1 0]  
[0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 0 0]  
[0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0 1 0]  
[1 0 0 1 1 0 0 1 0 0 0 1 0 0 1 1 1 1]  
[0 1 0 1 0 1 0 0 1 0 0 0 1 0 1 1 1 1]  
[0 0 1 0 1 1 0 0 0 1 0 0 0 1 1 1 1 1]  
True
```

```
version()
```

```
'SageMath version 7.3, Release Date: 2016-08-04'
```

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